Solutions to Quiz 4

1. Let $X(x) = \sin(3x)$. Then, $u(x,t) = X(x)T(t)$ is a solution to the wave equation $u_{tt} = 4u_{xx}$ if

   A. $\cosh(2t)$

   B. $\sin(3t)$

   C. None of these

   D. $\sinh(6t)$

   E. $\cos(6t)$

> **Solution:** Since $u(x,t) = \sin(3x)T(t)$ and $u_{tt} = 4u_{xx}$ yields
>
> $$4\frac{X''}{X} = -36 = \frac{T''}{T},$$
>
> $T(t)$ should satisfy $T'' + 36T = 0$. The general solution is $T(t) = C_1\cos(6t) + C_2\sin(6t)$ so that $\cos(6t)$ is the answer.

2. The function

$$u(x,t) = -5e^{-18t}\cos(3x) - e^{-50t}\sin(5x) + 9x + 7$$

   is a solution to

   A. $2u_{xx} = u_t$

   B. $2u_{xx} = u_t + 9x + 7$

   C. $u_{xx} = u_t$

   D. $u_{xx} = u_{tt}$

   E. $u_{xx} = u_t + 7$

> **Solution:** Computing u_{xx} and u_t, we get
>
> $$\begin{aligned} u_{xx} &= 45e^{-18t}\cos(3x) + 25e^{-50t}\sin(5x) \\ u_t &= 90e^{-18t}\cos(3x) + 50e^{-50t}\sin(5x). \end{aligned}$$
>
> Thus, we have $2u_{xx} = u_t$.

3. The solution to the heat conduction problem

$$\begin{aligned} 16u_{xx} &= u_t, \qquad \text{for } 0 < x < 5, \quad t > 0, \\ u(0,t) &= u(5,t) = 0, \qquad \text{for } t \geq 0, \\ u(x,0) &= 7\sin\left(\frac{\pi x}{5}\right) + 10\sin(\pi x), \end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{16n^2\pi^2 t}{25}} \sin\left(\frac{n\pi x}{5}\right).$$

Find C_1 and C_2.

Solution: Note that

$$C_n = \frac{2}{5}\int_0^5 \left(7\sin\left(\frac{\pi x}{5}\right) + 10\sin(\pi x)\right)\sin\left(\frac{n\pi}{5}x\right)\,dx.$$

By the orthogonality of sine functions, we have

$$\begin{aligned}\frac{2}{5}\int_0^5 7\sin\left(\frac{\pi x}{5}\right)\sin\left(\frac{n\pi}{5}x\right)\,dx &= \frac{7}{5}\int_{-5}^5 \sin\left(\frac{\pi x}{5}\right)\sin\left(\frac{n\pi}{5}x\right)\,dx \\ &= \begin{cases} 7, & n=1, \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\begin{aligned}\frac{2}{5}\int_0^5 10\sin(\pi x)\sin\left(\frac{n\pi}{5}x\right)\,dx &= \frac{1}{5}\int_{-5}^5 10\sin(\pi x)\sin\left(\frac{n\pi}{5}x\right)\,dx \\ &= \begin{cases} 10, & n=5, \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore, we obtain

$$C_n = \begin{cases} 7, & n=1, \\ 10, & n=5, \\ 0, & \text{otherwise.}\end{cases}$$

In particular, $C_1 = 7$ and $C_2 = 0$.

4. Find the solution $u(x,t)$ where $0 < x < L$ and $t > 0$ of the diffusion equation $u_t = \kappa\, u_{xx}$ $u(0,t) = T_1$; $\quad u(L,t) = T_2$ $u(x,0) = a$

   A. $u(x,t) = \sum_{m=1}^{\infty} \frac{2a}{m\pi}(1-(-1)^m)e^{-\frac{m^2\pi^2}{L^2}t}\sin(\frac{m\pi x}{L}) + \frac{(T_2-T_1)}{L}x + T_1$

   B. $u(x,t) = \sum_{m=1}^{\infty} a\frac{(m\pi)^2}{L}\sin(\frac{m\pi}{L}x)e^{-\frac{m^2\pi^2}{L}t} + T_2$

   C. $u(x,t) = \sum_{m=1}^{\infty} a\frac{(m\pi)^2}{L}\cos(\frac{m\pi}{L}x)e^{-\frac{m^2\pi^2}{L^2}t} + (T2-T1)/Lx + T_1$

   D. $u(x,t) = \sum_{m=1}^{\infty} a(-1)^m\sin(\frac{m\pi}{L}x)e^{-\frac{m^2\pi^2}{L^2}t}$

E. $u(x,t) = \sum_{m=1}^{\infty} a\sin(\frac{m\pi}{L}x)e^{-\frac{m^2\pi^2}{L^2}t}$

Solution:
The solution to the heat conduction problem $u_t = \kappa\, u_{xx}$ $u(0,t) = T_1; \qquad u(L,t) = T_2$ $u(x,0) = a$ is given by $u(x,t) = \sum_{m=1}^{\infty} b_n e^{-\frac{m^2\pi^2}{L^2}t}\sin(\frac{m\pi x}{L}) + v(x)$ where $b_m = \frac{2}{L}\int_0^L f(x)\sin(\frac{m\pi x}{L})dx == \frac{2}{L}\int_0^L a\left(\frac{L}{m\pi}(-\cos(\frac{m\pi x}{L})\mid_0^L\right) = \frac{2a}{m\pi}(1-(-1)^m)$ are the sine Fourier coefficients of $f(x) = a$ and $v(x)$ is the solution to

$$v'' = 0, \qquad v(0) = T_1; \qquad v(L) = T_2$$

so $v(x) = \frac{(T_2 - T_1)}{L}x + T_1$

5. Find the steady state solution $v(x)$ where $0 < x < L$ of the diffusion equation $u_t = \kappa\, u_{xx}$ $u(0,t) = B; \qquad u_x(L,t) = A$ $u(x,0) = a\,x^3$

   A. $v = Ax + B$

   B. $v = ax^3$

   C. $v = ax^3 + Ax + B$

   D. $v = e^{-\frac{n^2\pi^2}{L}t}$

   E. $v = \sum_{m=1}^{\infty} a\sin(\frac{m\pi}{L}x)e^{-\frac{n^2\pi^2}{L}t} + Ax + B$

Solution: The steady state solution satisfies

$$v'' = 0, \qquad v(0) = A; \qquad v'(L) = B$$

From $v'' = 0$ we get that v is linear $v = ax + b$ and plugging in the initial conditions

$$B = v(0) = b \qquad A = v'(L) = a$$

whence $v = Ax + B$

6. Write the solution $u(x,t)$ of

$$\begin{cases} u_{tt} = a^2 u_{xx}, & \text{for } 0 < x < L, \quad t > 0, \\ u(0,t) = 0, & \text{for } t \geq 0, \\ u(L,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = \sin\left(\frac{2\pi x}{L}\right) & \text{for } 0 \leq x \leq L \\ u_t(x,0) = 0, & \text{for } 0 \leq x \leq L \end{cases}$$

Solution: The solution is

$$u(x,t) = \sum_{m=1}^{\infty} b_n \sin(\frac{m\pi x}{L}) \cos(\frac{m\pi t}{L})$$

with $b_n = \frac{2}{L}\int_0^L \ f(x)\sin(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L \ \sin(\frac{2\pi x}{L})\sin(\frac{n\pi x}{L})dx$. Since the scalar product

$$\frac{2}{L}\int_0^L \ \sin(\frac{2\pi x}{L})\sin(\frac{n\pi x}{L})dx = 0 \qquad \text{for} \ . \ n \neq 2$$

and

$$\frac{2}{L}\int_0^L \ \sin(\frac{2\pi x}{L})\sin(\frac{2\pi x}{L})dx = 1$$

we have that $b_n = 0$ for $n \neq 2$ and $b_2 = 1$ which yields

$$u = \sin(2\pi x/L)\cos(2\pi t/L)$$