Math 285 Lecture Note: Week 7

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Lecture 16. Mechanical and Electrical Vibrations, Part 1 (Sec 3.7)

As an application, we study a spring–mass system. Consider a mass m hanging at rest on the end of a vertical spring. Hooke's law says that the spring force is proportional to the elongation L of the spring. Let $k > 0$ be the spring constant. Thus, in the equilibrium status, the gravity is the same as the spring force, that is $mg = kL$. Let $u(t)$ be the displacement of the mass from its equilibrium. Then the total force F is

$$F = mu''(t) = mg - k(L + u(t)) = -ku(t).$$

The characteristic equation is $m\lambda^2 + k = 0$. Thus, the general solution is

$$u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

where $\omega_0 = \sqrt{k/m}$ is the positive root of the characteristic equation and A, B are the constants determined by the initial conditions. To visualized the solution, it is good to reformulate it as

$$\begin{aligned} u(t) &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(\omega_0 t) + \frac{B}{\sqrt{A^2+B^2}}\sin(\omega_0 t)\right) \\ &= R\left(\cos\delta\cos(\omega_0 t) + \sin\delta\sin(\omega_0 t)\right) \\ &= R\cos(\omega_0 t - \delta) \end{aligned}$$

where $R = \sqrt{A^2+B^2}$ and $\delta \in [0, 2\pi)$ is the unique angle satisfying

$$\cos\delta = \frac{A}{\sqrt{A^2+B^2}}, \qquad \sin\delta = \frac{B}{\sqrt{A^2+B^2}}.$$

We call ω_0 the natural frequency, R the amplitude, and δ the phase. The period is $T = 2\pi/\omega_0$

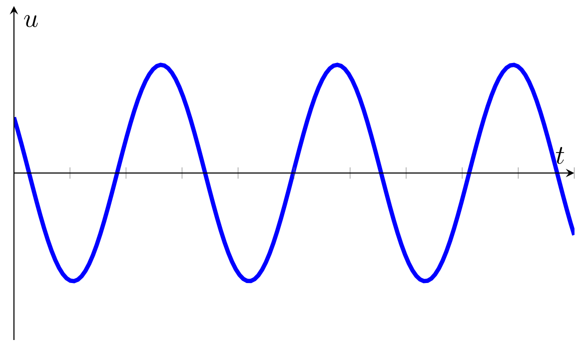
Example 1. Let $m = 2$ and $k = 6$. Suppose the initial conditions are $u(0) = 1$ and $u'(0) = -3$. Then, the natural frequency is $\omega_0 = \sqrt{k/m} = \sqrt{3}$, the period is $T = 2\pi/\sqrt{3}$, and the general solution is

$$u(t) = A\cos(\sqrt{3}t) + B\sin(\sqrt{3}t).$$

By the initial condition, we have

$$u(0) = A = 1, \qquad u'(0) = \sqrt{3}B = -3,$$

which yields $u(t) = \cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t)$. The amplitude is $R = \sqrt{1+3} = 2$ and the phase is $\delta = 5\pi/3$.



Lecture 17. Mechanical and Electrical Vibrations, Part 2 (Sec 3.7)

We include the affect of the resistive (or damping) force, which is proportional to the speed of the mass and its direction is the opposite of the movement of the mass. Thus, the total force is

$$F = mu''(t) = -ku(t) - \gamma u'(t)$$

where $\gamma > 0$ is called the damping constant. The characteristic equation is

$$m\lambda^2 + \gamma\lambda + k = 0.$$

By the quadratic formula, the roots for the equation is

$$\lambda = -\frac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2 - 4mk}{4m^2}} = -\frac{\gamma}{2m} \pm \sqrt{D}$$

where $D = \frac{\gamma^2 - 4mk}{4m^2}$.

If $D > 0$, then the characteristic equation has two distinct real roots $r_1, r_2 = -\frac{\gamma}{2m} \pm \sqrt{D} < 0$ and the general solution is

$$u(t) = Ae^{r_1 t} + Be^{r_2 t} = e^{-\frac{\gamma}{2m}t}(Ae^{\sqrt{D}t} + Be^{-\sqrt{D}t}).$$

This motion is called overdamped.

If $D = 0$, then the characteristic equation has repeated roots $-\frac{\gamma}{2m}$ and the general solution is

$$u(t) = e^{-\frac{\gamma}{2m}t}(At + B).$$

This motion is called critically damped.

If $D < 0$, then the characteristic equation has complex roots $r_1 = -\frac{\gamma}{2m} + i\mu$ and $r_2 = -\frac{\gamma}{2m} - i\mu$ where

$$\mu = \sqrt{-D} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$$

and the general solution is

$$u(t) = e^{-\frac{\gamma}{2m}t}(A\cos\mu t + B\sin\mu t).$$

We call μ the quasi frequency and $T_d = 2\pi/\mu$ the quasi period. This motion is called underdamped.

Example 2. Consider the spring–mass system with damping

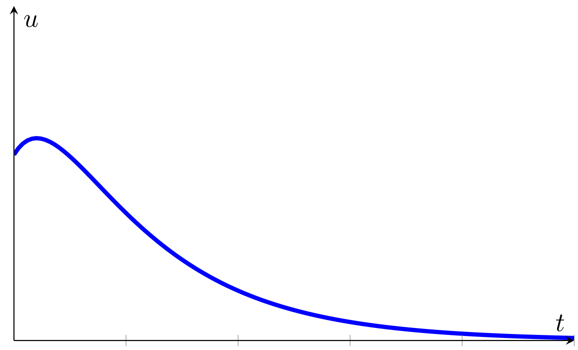
$$mu'' + \gamma u' + ku = 0$$

with $m = 1$, $\gamma = 5$, $k = 4$ and the initial conditions $u(0) = 1$ and $u'(0) = 1$. Then, the general solution is

$$u(t) = Ae^{-t} + Be^{-4t}.$$

By the initial condition, we have

$$u(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}.$$



Example 3. Consider the spring–mass system with damping

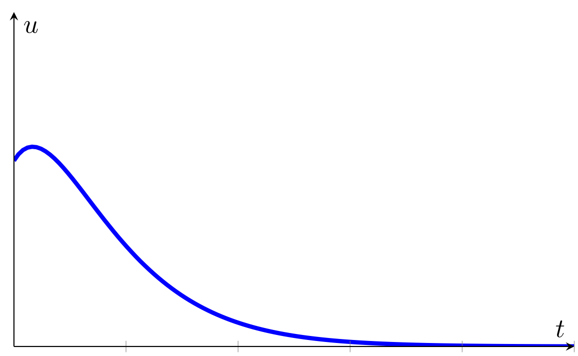
$$mu'' + \gamma u' + ku = 0$$

with $m = 1$, $\gamma = 4$, $k = 4$ and the initial conditions $u(0) = 1$ and $u'(0) = 1$. Then, the general solution is

$$u(t) = (At + B)e^{-2t}.$$

By the initial condition, we have

$$u(t) = (3t + 1)e^{-2t}.$$



Example 4. Consider the spring–mass system with damping

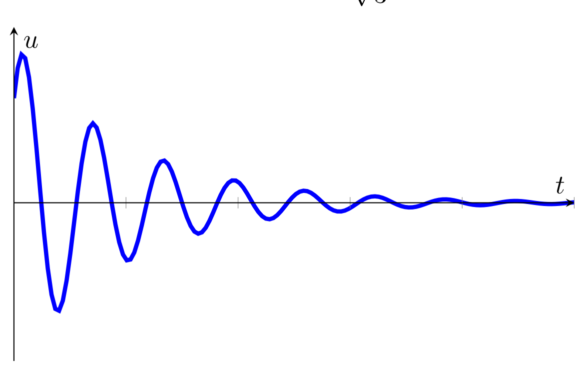
$$mu'' + \gamma u' + ku = 0$$

with $m = 1$, $\gamma = 2$, $k = 4$ and the initial conditions $u(0) = 1$ and $u'(0) = 1$. Then, the general solution is

$$u(t) = e^{-t}(A\cos\sqrt{3}t + B\sin\sqrt{3}t).$$

By the initial condition, we have

$$u(t) = e^{-t}(\cos\sqrt{3}t + \frac{2}{\sqrt{3}}\sin\sqrt{3}t).$$



Lecture 18. Forced Vibrations without damping (Sec 3.8)

In this section, we investigate the spring–mass system with external forces.

We start with forced vibrations without damping effect. Suppose there is an external force given by $F_0 \cos(\omega t)$, then we have

$$mu''(t) + ku(t) = F_0 \cos(\omega t)$$

where F_0 is a constant. Suppose $\omega \neq \omega_0$. The general solution is

$$u(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

If $u(0) = u'(0) = 0$, then

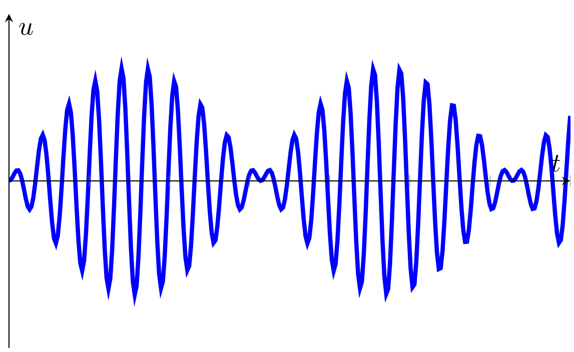
$$\begin{aligned} u(t) &= \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t) \\ &= \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left(\frac{(\omega_0 + \omega)}{2} t \right) \sin \left(\frac{(\omega_0 - \omega)}{2} t \right). \end{aligned}$$

Example 5. Consider the case where $m = 1$, $k = 4$, $F_0 = 3$, and $\omega = 1.8$. Then,

$$u''(t) + 4u(t) = 3 \cos(1.8t).$$

Then $\omega_0 = 2$ and the solution with initial conditions $u(0) = u'(0) = 0$ is

$$\begin{aligned} u(t) &= \frac{3}{(4 - 3.24)} (\cos 1.8t - \cos 2t) \\ &= \frac{6}{0.76} \sin (1.9t) \sin (0.1t) \end{aligned}$$



What if $\omega_0 = \omega$? In this case, the general solution is

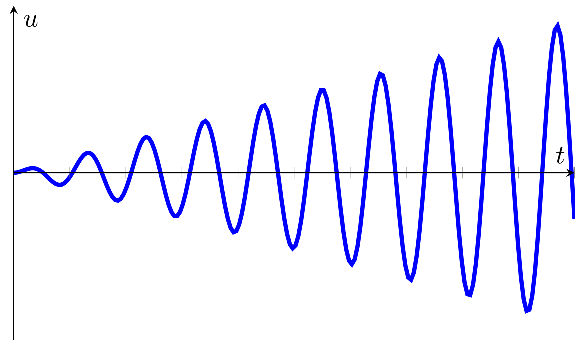
$$u(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.$$

Example 6. Consider

$$u'' + u = 3 \cos t, \qquad u(0) = u'(0) = 0.$$

Then,

$$u(t) = 1.5t \sin t.$$



References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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