Math 285 Lecture Note: Week 6

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Lecture 13. Repeated Roots; Reduction of Order (Sec 3.4)

Consider $y'' + 6y' + 9y = 0$, then the characteristic equation is $r^2 + 6r + 9 = 0$. Thus, it has the repeated root $r = -3$ and so $y_1(t) = e^{-3t}$ is a solution. In this case, the characteristic equation only gives one solution, which is not enough for finding the general solution.

To find the general solution, we need to find another solution which is different enough from $y_1(t)$. Since $y_1(t)$ is a solution, so is $cy_1(t)$ for any constant c. But the problem is that this solution is not different enough from $y_1(t)$. That is, $W[y_1, cy_1](t) = 0$ for all t. The idea is to replace the constant c with a function $v(t)$. Let $y(t) = v(t)y_1(t) = v(t)e^{-3t}$. If it is a solution, then

$$\begin{aligned}
y'' + 6y' + 9y &= (v(t)e^{-3t})'' + 6(v(t)e^{-3t})' + 9(v(t)e^{-3t}) \\
&= (v''(t)e^{-3t} - 6v'(t)e^{-3t} + 9v(t)e^{-3t}) + 6(v'(t)e^{-3t} - 3v(t)e^{-3t}) + 9(v(t)e^{-3t}) \\
&= e^{-3t}v''(t) \\
&= 0.
\end{aligned}$$

Thus, $v'' = 0$ and so $v(t) = C_1 + C_2 t$. Therefore,

$$y(t) = C_1 e^{-3t} + C_2 t e^{-3t}.$$

Let $u = e^{-3t}$ and $v = te^{-3t}$, then the Wronskian is

$$\begin{aligned}
W[u, v](t) &= uv' - u'v \\
&= e^{-3t}(e^{-3t} - 3te^{-3t}) + 3te^{-6t} \\
&= e^{-6t}.
\end{aligned}$$

Thus, $\{u, v\}$ is a fundamental set of solutions. In general, if the characteristic equation has the repeat root r_1, then the general solution is

$$y(t) = C_1 e^{r_1 t} + C_2 t e^{r_1 t}.$$

Example 1. Consider $4y'' - 12y' + 9y = 0$ with $y(0) = 6$ and $y'(0) = 3$. Since the characteristic equation $4r^2 - 12r + 9 = 0$ has the repeated root $r_1 = 3/2$, the general solution is

$$y(t) = C_1 e^{3t/2} + C_2 t e^{3t/2}.$$

Since

$$y'(t) = \frac{3}{2} C_1 e^{3t/2} + C_2 e^{3t/2} + \frac{3}{2} C_2 t e^{3t/2},$$

we have $y(0) = \frac{3}{2}C_1 = 6$ and $y'(0) = \frac{3}{2}C_1 + C_2 = 3$, which gives

$$y(t) = 6e^{3t/2} - 6te^{3t/2}.$$

Consider $y'' + py' + qy = 0$ for $p, q \in \mathbb{R}$. Suppose r_1, r_2 are the roots of the characteristic equation $r^2 + pr + q = 0$. Then, the general solution is

$$y(t) = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & \text{if } r_1, r_2 \in \mathbb{R}, r_1 \neq r_2, \\ e^{\lambda t}(C_1 \cos \mu + C_2 \sin \mu), & \text{if } r_1, r_2 \in \mathbb{C}, r_1 = \lambda + i\mu, r_2 = \lambda - i\mu, \\ C_1 e^{r_1 t} + C_2 t e^{r_1 t}, & \text{if } r_1 = r_2. \end{cases}$$

Example 2. Consider $t^2 y'' + 3ty' + y = 0$ for $t > 0$. One can see that $y_1(t) = t^{-1}$ is a solution. Our goal is to find another solution from y_1 using the method that we used before. Let $y(t) = v(t)y_1(t) = v(t)t^{-1}$. If it is a solution, then

$$\begin{aligned} t^2 y'' + 3ty' + y &= t^2(v(t)t^{-1})'' - 4t(v(t)t^{-1})' + 6v(t)t^{-1} \\ &= t^2(v''(t)t^{-1} - 2v'(t)t^{-2} + 2v(t)t^{-3}) + 3t(v'(t)t^{-1} - v(t)t^{-2}) + v(t)t^{-1} \\ &= tv''(t) + v' \\ &= 0. \end{aligned}$$

Let $w = v'$, then we obtain a first order ODE: $tw' = -w$. Since it is separable, we solve it to obtain

$$w = \frac{C}{t} = v'.$$

Thus, $v(t) = C_1 + C_2 \ln t$ and the general solution is

$$y(t) = \frac{C_1}{t} + C_2 \frac{\ln t}{t}.$$

This method is called the reduction of the order.

Lecture 14. Nonhomogeneous Equations; Method of Undetermined Coefficients (Sec 3.5)

Let $p(t), q(t), g(t)$ are continuous functions on an open interval I. Let L be a differential operator defined by $L[y] = y'' + p(t)y' + q(t)y$. Consider a nonhomogeneous equation $L[y] = g(t)$ and the corresponding homogeneous equation $L[y] = 0$.

Theorem 3. *Let Y_1 and Y_2 be solutions to $L[y] = g(t)$, then $Y_1 - Y_2$ is a solution to $L[y] = 0$.*

Theorem 4. *Let Y be a solution to $L[y] = g(t)$ and $\{y_1, y_2\}$ a fundamental set of solutions of $L[y] = 0$, then the general solution to $L[y] = g(t)$ is*

$$y(t) = Y(t) + C_1 y_1(t) + C_2 y_2(t).$$

In this section, we discuss how we can find a particular solution to nonhomogeneous equations where $g(t)$ is a sum of products of polynomials, exponentials, and trigonometric functions.

Theorem 5. *Consider $L[y] = g(t)$ and let $g(t) = g_1(t) + g_2(t)$. Let Y_1, Y_2 be solutions to $L[y] = g_1(t)$ and $L[y] = g_2(t)$ respectively. Then $Y_1 + Y_2$ is a solution to $L[y] = g(t)$.*

1 Polynomials

Example 6. Consider $y'' - 2y' - 3y = 3t^2$. Let $Y(t) = At^2 + Bt + C$, then

$$\begin{aligned} y'' - 2y' - 3y &= (At^2 + Bt + C)'' - 2(At^2 + Bt + C)' - 3(At^2 + Bt + C) \\ &= 2A - 4At - 2B - 3At^2 - 3Bt - 3C \\ &= -3At^2 - (4A + 3B)t + (2A - 2B - 3C) \\ &= 3t^2. \end{aligned}$$

Thus, $A=-1$, $B=4/3$, and $C=-14/9$. Thus, the general solution is

$$y(t) = -t^2 + \frac{4}{3}t - \frac{14}{9} + C_1 e^{-t} + C_2 e^{3t}.$$

If $y'' - 2y' = 3t^2$, then $Y(t) = At^2 + Bt + C$ won't work because the degree of the LHS is 1. In this case, $Y(t) = t(At^2 + Bt + C)$. If $y'' = 3t^2$, then $Y(t) = t^2(At^2 + Bt + C)$ will work. In general, if $py'' + qy' + ry = P_n(t)$ and $P_n(t)$ is a polynomial of degree n, then

$$Y(t) = \begin{cases} A_n t^n + \cdots + A_0, & \text{if } r \neq 0, \\ t(A_n t^n + \cdots + A_0), & \text{if } r = 0, q \neq 0, \\ t^2(A_n t^n + \cdots + A_0), & \text{if } r = q = 0. \end{cases}$$

2 Exponential functions

Example 7. Consider $y'' - 2y' - 3y = 2e^t$. Let $Y(t) = Ce^t$, then $-4Ce^t = 2e^t$ and so $C = -\frac{1}{2}$. Thus, the general solution is

$$y(t) = -\frac{1}{2}e^t + C_1 e^{-t} + C_2 e^{3t}.$$

Example 8. Consider $y'' - 2y' - 3y = 4e^{-t}$. Let $Y(t) = Ce^{-t}$, then $0 = 4e^{-t}$. This is because Ce^{-t} is a solution to the corresponding homogeneous equation. Let $Y(t) = Cte^{-t}$, then

$$\begin{aligned} y'' - 2y' - 3y &= (Cte^{-t})'' - 2(Cte^{-t})' - 3Cte^{-t} \\ &= C(te^{-t} - 2e^{-t} - 2e^{-t} + 2te^{-t} - 3te^{-t}) \\ &= -4Ce^{-t} \\ &= 4e^{-t}. \end{aligned}$$

So $C = -1$ and $Y(t) = -te^{-t}$. Thus, the general solution is

$$y(t) = -te^{-t} + C_1 e^{-t} + C_2 e^{3t}.$$

Example 9. Consider $y'' - 2y' + y = 2e^t$. As before $Y(t) = Ce^t$ won't work because it is a solution to the corresponding homogeneous equation. However, $Y(t) = Cte^t$ is also a solution because the characteristic has the repeated solution $r = 1$. In this case, we try $Y(t) = Ct^2e^t$. Indeed, we have

$$\begin{aligned} y'' - 2y' + y &= (Ct^2e^t)'' - 2(Ct^2e^t)' + Ct^2e^t \\ &= C(t^2e^t + 4te^t + 2e^t - 4te^t - 2t^2e^t + t^2e^t) \\ &= 2Ce^t \\ &= 2e^t. \end{aligned}$$

Then $2Ce^t = 2e^t$ and so $C = 1$. Thus, the general solution is

$$y(t) = t^2e^t + C_1 e^{-t} + C_2 e^{3t}.$$

In general, if $py'' + qy' + ry = P_n(t)e^{\alpha t}$ and $P_n(t)$ is a polynomial of degree n, then

$$Y(t) = \begin{cases} e^{\alpha t}(A_n t^n + \cdots + A_0), & \text{or} \\ te^{\alpha t}(A_n t^n + \cdots + A_0), & \text{or} \\ t^2e^{\alpha t}(A_n t^n + \cdots + A_0). \end{cases}$$

This depends on whether $e^{\alpha t}$ and $te^{\alpha t}$ are solutions to the corresponding homogeneous equation.

3 Sine and cosine functions

Example 10. Consider $y'' - 2y' - 3y = 13\sin(2t)$. Let $Y(t) = A\cos(2t) + B\sin(2t)$, then

$$\begin{aligned} y'' - 2y' - 3y &= -4(A\cos(2t) + B\sin(2t)) + 4(A\sin(2t) - B\cos(2t)) - 3(A\cos(2t) + B\sin(2t)) \\ &= -(7A + 4B)\cos(2t) + (4A - 7B)\sin(2t) \\ &= 13\sin(2t). \end{aligned}$$

So, $A = -7/5$ and $B = -4/5$. Thus, the general solution is

$$y(t) = -\frac{1}{5}(7\cos(2t) + 4\sin(2t)) + C_1 e^{-t} + C_2 e^{3t}.$$

In general, if $py'' + qy' + ry = g(t)$ where $g(t) = P_n(t)e^{\alpha t}\cos(\beta t)$ or $g(t) = P_n(t)e^{\alpha t}\sin(\beta t)$ and $P_n(t)$ is a polynomial of degree n, then

$$Y(t) = t^s(e^{\alpha t}(A_n t^n + \cdots + A_0)\cos(\beta t) + e^{\alpha t}(A_n t^n + \cdots + A_0)\sin(\beta t))$$

for some $s = 0, 1, 2$.

Lecture 15. Variation of Parameters (Sec 3.6)

Theorem 11. *Let $p(t), q(t), g(t)$ are continuous on an open interval I and $L[y] = y'' + p(t)y' + q(t)y$. Let $\{y_1, y_2\}$ be a fundamental set of solutions to $L[y] = 0$ and $t_0 \in I$, then*

$$Y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$$

is a particular solution to $L[y] = g(t)$ where

$$\begin{aligned} v_1(t) &= -\int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\, ds, \\ v_2(t) &= \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\, ds. \end{aligned}$$

Proof. The idea is to let $Y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ and find the conditions for $v_1(t)$ and $v_2(t)$. Assume that $v_1'y_1 + v_2'y_2 = 0$ for simplicity (because we will determine v_1, v_2 later), then

$$Y' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' = v_1y_1' + v_2y_2'$$

and

$$Y'' = v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'.$$

If Y is a solution to the equation $L[y] = g(t)$, then

$$\begin{aligned} L[Y] &= v_1L[y_1] + v_2L[y_2] + v_1'y_1' + v_2'y_2' \\ &= v_1'y_1' + v_2'y_2' \\ &= g(t). \end{aligned}$$

Thus, our goal is to find v_1 and v_2 such that

$$\begin{aligned} v_1'y_1 + v_2'y_2 &= 0, \\ v_1'y_1' + v_2'y_2' &= g(t). \end{aligned}$$

This can be written in terms of matrices:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

Solving this system of equations, we get

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \frac{1}{W[y_1, y_2]} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(t) \end{pmatrix} = \frac{1}{W[y_1, y_2](t)} \begin{pmatrix} -y_2(t)g(t) \\ y_1(t)g(t) \end{pmatrix}.$$

□

Example 12. Consider $y'' - 2y' - 3y = 2e^t$, then $y_1(t) = e^{-t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. Direct computations yield

$$\begin{aligned}
W[y_1, y_2](t) &= y_1 y_2' - y_1' y_2 = 4e^{2t} \\
v_1(t) &= -\int_0^t \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\, ds \\
&= -\frac{1}{2}\int_0^t e^{2s}\, ds \\
&= -\frac{1}{4}(e^{2t} - 1) \\
v_2(t) &= \int_0^t \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\, ds \\
&= \frac{1}{2}\int_0^t e^{-2s}\, ds \\
&= -\frac{1}{4}(e^{-2t} - 1).
\end{aligned}$$

Thus, we obtain a particular solution

$$\begin{aligned}
Y(t) &= v_1(t)y_1(t) + v_2(t)y_2(t) \\
&= -\frac{1}{4}(e^{2t} - 1)e^{-t} - \frac{1}{4}(e^{-2t} - 1)e^{3t} \\
&= -\frac{1}{2}e^t + \frac{1}{4}(e^{-t} + e^{3t}).
\end{aligned}$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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