Math 285 Lecture Note: Week 5

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Lecture 12. Complex Roots of the Characteristic Equation (Sec 3.3)

Example 1. Consider $y'' + 2y' + 5 = 0$. As before, let $y(t) = e^{rt}$, then we get the characteristic equation $r^2+2r+5 = 0$. The roots for the equation are $r = -1+2i, -1-2i$. This yields that $y(t) = e^{(-1+2i)t}, e^{(-1-2i)t}$ are solutions to the DE. What are these functions?

Example 2. Let's consider $y'' + y = 0$, then the characteristic equation is $r^2 + 1 = 0$, whose roots are $r = i, -i$. Thus, $y(t) = e^{it}, e^{-it}$ are solutions. On the other hands, we already know that $y(t) = \sin t, \cos t$ are solutions.

Euler's Formula enables us to define the exponent with complex numbers, which says

$$e^{it} = \cos t + i \sin t.$$

For example, $e^{\pi i/2} = i$, $e^{\pi i} = -1$, and $e^{\pi i/3} = \cos(\pi/3) + i \sin(\pi/3) = \frac{1}{2}(1 + \sqrt{3}i)$. For general complex number $z = x + iy$, we have

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Note that if z is a complex number, then so is e^z and it could be negative.

Example 3. Consider $y'' + 2y' + 5 = 0$, then $y(t) = e^{r_1 t}, e^{r_2 t}$ are solutions where $r_1 = -1 + 2i$ and $r_2 = -1 - 2i$. By the previous argument, we have

$$\begin{aligned}
y_1(t) &= e^{-t}(\cos(2t) + i \sin(2t)),\\
y_2(t) &= e^{-t}(\cos(-2t) + i \sin(-2t)) = e^{-t}(\cos(2t) - i \sin(2t)).
\end{aligned}$$

Then, the Wronskian is

$$\begin{aligned}
W[y_1, y_2](t) &= \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}\\
&= r_2 y_1(t) y_2(t) - r_1 y_1(t) y_2(t)\\
&= (r_2 - r_1)e^{(r_1+r_2)t}\\
&= -4ie^{-2t}
\end{aligned}$$

Since $W[y_1, y_2](0) = -4i \neq 0$, $\{y_1, y_2\}$ is a fundamental set of solutions.

However, this fundamental set is not convenient because y_1 and y_2 are complex-valued solutions. Consider

$$\begin{aligned}
u(t) &= \frac{1}{2}(y_1(t) + y_2(t)) = e^{-t} \cos(2t),\\
v(t) &= \frac{1}{2i}(y_1(t) - y_2(t)) = e^{-t} \sin(2t).
\end{aligned}$$

Note that these are also solutions. Furthermore, the Wronskian is

$$\begin{aligned} W[u,v](t) &= u(t)v'(t) - u'(t)v(t) \\ &= e^{-2t}(\cos(2t)(2\cos(2t) - \sin(2t)) + \sin(2t)(\cos(2t) + 2\sin(2t))) \\ &= 2e^{-2t}(\cos^2(2t) + \sin^2(2t)) \\ &= 2e^{-2t} \\ &\neq 0. \end{aligned}$$

Thus, $\{u, v\}$ is a fundamental set of solutions. The general real valued solution of the DE is

$$y(t) = e^{-t}(C_1 \cos(2t) + C_2 \sin(2t)).$$

Note that

$$\begin{aligned} W[u,v](t) &= u(t)v'(t) - u'(t)v(t) \\ &= \frac{1}{4i}((y_1 + y_2)(y_1' - y_2') - (y_1' + y_2')(y_1 - y_2)) \\ &= -\frac{1}{2i}(y_1 y_2' - y_1' y_2) \\ &= -\frac{1}{2i} W[y_1, y_2](t). \end{aligned}$$

This is because

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}.$$

Example 4. Consider $y'' - 4y' + 5y = 0$ with $y(0) = 1$ and $y'(0) = 3$. The characteristic equation is $r^2 - 4r + 5 = 0$ and so the roots are $r = 2 + i, 2 - i$. Thus, $\{u, v\}$ is a fundamental set of solutions where $u = e^{2t} \cos t$ and $v = e^{2t} \sin t$. The general solution is

$$y(t) = e^{2t}(C_1 \cos t + C_2 \sin t).$$

Note that

$$y'(t) = 2e^{2t}(C_1 \cos t + C_2 \sin t) + e^{2t}(-C_1 \sin t + C_2 \cos t).$$

By the initial conditions, we have

$$\begin{aligned} y(0) &= C_1 = 1, \\ y'(0) &= 2C_1 + C_2 = 3, \end{aligned}$$

and $y(t) = e^{2t}(\cos t + \sin t)$.

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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