Math 285 Lecture Note: Week 16

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Lecture 36. Sturm–Liouville Boundary Value Problems (Sec 11.2)

We recall the definition of eigenvalues and eigenfunctions for a general differential operator.

Definition 1. Let $L[y]$ be a differential operator. We say that λ is a real (or complex) eigenvalue of L with a homogeneous boundary conditions if $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) and the differential equation $L[y] = \lambda y$ with the given homogeneous boundary conditions has nontrivial solutions. The nontrivial solutions are called eigenfunctions.

In this section, we study the Sturm–Liouville boundary problem, which consists of a differential equation of the form

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

on $[0, 1]$ with boundary conditions

$$\begin{aligned} \alpha_1 y(0) + \alpha_2 y'(0) &= 0, \\ \beta_1 y(1) + \beta_2 y'(1) &= 0 \end{aligned}$$

with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$. We further assume that p, p', q, r are continuous on $[0, 1]$ and $p(x), r(x) > 0$ for all $x \in [0, 1]$. In this case, the problem is called *regular*.

Example 2. If $p(x) = r(x) = 1$ and $q(x) = 0$, then the problem is $y'' + \lambda y = 0$ with

$$\begin{aligned} \alpha_1 y(0) + \alpha_2 y'(0) &= 0, \\ \beta_1 y(1) + \beta_2 y'(1) &= 0. \end{aligned}$$

Example 3. If $p(x) = x^k$, $q(x) = 0$, and $r(x) = x^{k-2}$, then

$$\begin{aligned} (p(x)y')' - q(x)y + \lambda r(x)y &= x^k y'' + kx^{k-1}y' + \lambda x^{k-2} y \\ &= x^{k-2}(x^2 y'' + kxy' + \lambda y) \\ &= 0. \end{aligned}$$

Thus, the problem $x^2 y'' + kxy' + \lambda y = 0$ with

$$\begin{aligned} \alpha_1 y(1) + \alpha_2 y'(1) &= 0, \\ \beta_1 y(2) + \beta_2 y'(2) &= 0. \end{aligned}$$

also belongs to the class.

Proposition 4 (Lagrange's identity)**.** *Let u and v be functions on $[0, 1]$ with continuous second derivatives and L a differential operator defined by $L[y] = -(p(x)y')' + q(x)y$. Then,*

$$\begin{aligned} \int_0^1 (L[u]v - uL[v])\, dx &= -p(x) \left[u'(x)v(x) - u(x)v'(x)\right]_0^1 \\ &= p(0)\left(u'(0)v(0) - u(0)v'(0)\right) - p(1)\left(u'(1)v(1) - u(1)v'(1)\right). \end{aligned}$$

Furthermore, if u and v satisfies the boundary condition

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0,$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0$$

with $\alpha_1^2 + \alpha_2^2 > 0$ *and* $\beta_1^2 + \beta_2^2 > 0$*, then*

$$\int_0^1 (L[u]v - uL[v])\, dx = 0.$$

Remark 5. The Lagrange's identity with the boundary condition can be written as

$$(L[u], v) = (u, L[v])$$

where the inner product on $[0, 1]$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)\, dx.$$

The Lagrange's identity also holds for complex-valued functions u and v with the complex inner product

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx.$$

Theorem 6. *Every eigenvalue of the Sturm–Liouville problem is real.*

Proof. Let λ be an eigenvalue and ϕ the corresponding eigenfunction. Since ϕ is a solution to the Sturm–Liouville problem , we have

$$\int_0^1 L[\phi]\overline{\phi}\, dx = \int_0^1 \phi\overline{L[\phi]}\, dx$$

and $L[\phi] = \lambda r(x)\phi$. Since $r(x) > 0$ and ϕ is nontrivial, we get

$$\int_0^1 L[\phi]\overline{\phi}\, dx = \lambda \int_0^1 r(x)\phi\overline{\phi}\, dx = \overline{\lambda} \int_0^1 r(x)\phi\overline{\phi}\, dx = \int_0^1 \phi\overline{L[\phi]}\, dx$$

and so $\lambda = \overline{\lambda}$. □

Theorem 7. *If* ϕ_n *and* ϕ_m *are two eigenfunctions of the Sturm–Liouville problem corresponding to distinct eigenvalues* λ_n *and* λ_m*, then*

$$\int_0^1 \phi_n(x)\phi_m(x)\, r(x)dx = 0.$$

We call ϕ_n *and* ϕ_m *are orthogonal on* $[0, 1]$ *with the weight* $r(x)$*.*

Proof. We have $L[\phi_n] = \lambda_n r(x)\phi_n$ and $L[\phi_m] = \lambda_m r(x)\phi_m$. It follows from the Lagrange's identity that

$$\begin{aligned}\int_0^1 L[\phi_n]\phi_m\, dx - \int_0^1 \phi_n L[\phi_m]\, dx &= \lambda_n \int_0^1 \phi_n(x)\phi_m(x)\, r(x)dx - \lambda_m \int_0^1 \phi_n(x)\phi_m(x)\, r(x)dx \\ &= (\lambda_n - \lambda_m) \int_0^1 \phi_n(x)\phi_m(x)\, r(x)dx \\ &= 0.\end{aligned}$$

Since $\lambda_n - \lambda_m \neq 0$, we obtain the desired result. □

Theorem 8. *Every eigenvalue of the Sturm–Liouville problem is simple; that is, if ϕ_1 and ϕ_2 are the corresponding eigenfunctions of the same eigenvalue λ, then they are linearly dependent. Furthermore, the eigenvalues form an infinite sequence $\lambda_1 < \lambda_2 < \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$.*

Definition 9. An eigenfunction ϕ is normalized with respect to the weight r if

$$\int_0^1 \phi^2 \, r(x)dx = 1.$$

Remark 10. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues and ϕ_n the corresponding normalized eigenfunctions. Then,

$$\int_0^1 \phi_n \phi_m \, r(x)dx = \begin{cases} 1, & m = n \\ 0 & m \neq n. \end{cases}$$

In this case, we call the set $\{\phi_n : n = 1, 2, \cdots\}$ is orthonormal.

Suppose that we are given a function $f(x)$ and want to represent it in terms of the normalized eigenfunctions ϕ_n as we did in the heat conduction equation. To be specific, the question is to find C_n such that

$$f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x).$$

Using the orthogonality and the normalization, it is obvious to guess that

$$C_n = \int_0^1 f(x)\phi_n(x) \, r(x)dx.$$

Theorem 11. *Let $\phi_1, \phi_2, \cdots$ be the normalized eigenfunctions of the Sturm–Liouville problem. If $f(x)$ and $f'(x)$ are piecewise continuous on $[0, 1]$, then*

$$\sum_{n=1}^{\infty} C_n \phi_n(x) = \frac{1}{2}(f(x+) + f(x-))$$

with

$$C_n = \int_0^1 f(x)\phi_n(x) \, r(x)dx.$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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