# Math 285 Lecture Note: Week 15

Daesung Kim

## Lecture 33. Laplace's Equation: Dirichlet problem for a rectangle (Sec 10.8)

Consider a 2-dimensional heat equation $\alpha^2(u_{xx}+u_{yy}) = u_t$. If a steady state temperature distribution exists, $u$ is a function of $x$ and $y$ and satisfies

$$u_{xx} + u_{yy} = 0.$$

This is called Laplace's equation. Since there is no time dependence, we do not have initial condition.

In 1-dimension, boundary conditions refer prescribed function values or derivatives at the ends of a given interval. In higher dimension, information at two points is not sufficient. In general, boundary conditions are conditions at all points of the boundary.

The problem of finding a solution of Laplace's equation with prescribed function values on the boundary is called a Dirichlet problem. The problem of finding a solution of Laplace's equation with prescribed normal derivatives on the boundary is called a Neumann problem.

In this section, we focus on 2-dimensional Dirichlet problems for a rectangle and a disk.

Consider

$$u_{xx} + u_{yy} = 0 \tag{1}$$

in the rectangle $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$ with

$$\begin{aligned} u(x,0) &= 0, \qquad u(x,b) = 0 \quad \text{for } 0 < x < a, \\ u(0,y) &= 0, \qquad u(a,y) = f(y) \quad \text{for } 0 \le y \le b, \end{aligned} \tag{2}$$

where $f$ is a function on $0 \le y \le b$. By the method of separation, we let $u(x, y) = X(x)Y(y)$ and have

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The boundary conditions (3) read

$$Y(0) = Y(b) = 0, \qquad X(0) = 0$$

Solving $Y'' + \lambda Y = 0$ with $Y(0) = Y(b) = 0$, we get

$$\lambda_n = \frac{n^2\pi^2}{b^2}$$
$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right).$$

For each $\lambda_n$, we solve $X'' - \lambda_n X = 0$ with $X(0) = 0$ to obtain

$$X_n(x) = \sinh\left(\frac{n\pi}{b}x\right)$$

and so

$$u(x,y) = \sum_{n=1}^{\infty} C_n u_n(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right).$$

Now, the condition $u(a,y) = f(y)$ implies

$$u(a,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right) \sin\left(\frac{n\pi}{b}y\right) = f(y).$$

Using the Fourier sine series of $f$, the constants $C_n$ are determined by

$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi}{b}y\right)\, dy.$$

**Example 1.** Consider $u_{xx} + u_{yy} = 0$ in the rectangle $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2\}$ with

$$\begin{aligned} &u(x,0) = 0, \qquad u(x,2) = 0 \quad \text{for } 0 < x < 2, \\ &u(0,y) = 0, \qquad u(2,y) = f(y) \quad \text{for } 0 \le y \le 2, \end{aligned} \tag{3}$$

where $f(y) = 2y - y^2$. Then, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{2}x\right) \sin\left(\frac{n\pi}{2}y\right)$$

where

$$C_n = \frac{1}{\sinh(n\pi)} \int_0^2 (2y - y^2) \sin\left(\frac{n\pi}{2}y\right)\, dy = \frac{16(1-(-1)^n)}{\pi^3 n^3 \sinh(n\pi)}.$$

# Lecture 34. Laplace's Equation: Dirichlet problem for a disk (Sec 10.8)

Consider the 2-dimensional Laplace's equation in the disk

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}.$$

with the boundary condition $u(a\cos\theta, a\sin\theta) = f(\theta)$ for $0 \le \theta < 2\pi$. For a disk, it is convenient to use polar coordinates. Recall that polar coordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

for $r > 0$ and $0 \le \theta < 2\pi$. Using this, the disk can be written as $\mathcal{D} = \{(r,\theta) : 0 \le r < a, 0 \le \theta < 2\pi\}$. We abuse the notation $u(x,y) = u(r,\theta)$, then the boundary condition can be written as $u(a,\theta) = f(\theta)$. We translate the Laplace's equation in terms of polar coordinates. By chain rule, we have

$$\begin{aligned} u_r &= \frac{\partial x}{\partial r}u_x + \frac{\partial y}{\partial r}u_y = (\cos\theta)u_x + (\sin\theta)u_y, \\ u_\theta &= \frac{\partial x}{\partial \theta}u_x + \frac{\partial y}{\partial \theta}u_y = (-r\sin\theta)u_x + (r\cos\theta)u_y, \\ u_{rr} &= \frac{\partial x}{\partial r}(u_r)_x + \frac{\partial y}{\partial r}(u_r)_y = (\cos^2\theta)u_{xx} + (2\cos\theta\sin\theta)u_{xy} + (\sin^2\theta)u_{yy}, \\ u_{\theta\theta} &= \frac{\partial x}{\partial \theta}(u_\theta)_x + \frac{\partial y}{\partial \theta}(u_\theta)_y \\ &= (r^2\sin^2\theta)u_{xx} + (-2r^2\sin\theta\cos\theta)u_{xy} + (r^2\cos^2\theta)u_{yy} + (-r\cos\theta)u_x + (-r\sin\theta)u_y \\ &= r^2(u_{xx} + u_{yy}) - r^2 u_{rr} - r u_r. \end{aligned}$$

So, if $u_{xx} + u_{yy} = 0$, then

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $u(r, \theta) = R(r)\Theta(\theta)$, then

$$\begin{aligned} r^2 R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) &= 0 \\ r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} &= -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda. \end{aligned}$$

Thus, we get

$$r^2 R'' + rR' - \lambda R = 0, \qquad \Theta'' + \lambda\Theta = 0.$$

We first solve the equation for $\Theta$. Note that we do not have any boundary conditions for $\Theta$. Instead, $\Theta$ is periodic with period $2\pi$.

Suppose $\lambda = -\mu^2 < 0$, then

$$\Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

Since $\Theta$ is periodic, $c_1$ and $c_2$ should be zero. That is, $\Theta = 0$.

Suppose $\lambda = 0$, then $\Theta(\theta) = c_1 + c_2\theta$. Due to the periodicity, $c_2 = 0$, that is, $\Theta$ is a constant. In this case, we have $rR'' + R' = 0$ and so the general solution is

$$R(r) = c_1 + c_2 \ln r.$$

If $r$ tends to 0, $\ln r$ diverges. Since we are interested in the case where $u$ is bounded in the disk $\mathcal{D}$, $c_2$ should be 0 and $R$ is also a constant. Thus, the fundamental solution corresponding to $\lambda = 0$ is $u_0(r, \theta) = 1$.

Suppose $\lambda = \mu^2 > 0$, then

$$\Theta(\theta) = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta).$$

Since $\Theta$ is periodic with period $2\pi$, $\mu$ should be a positive integer. For each $n \in \mathbb{N}$, we want to solve $r^2 R'' + rR' - n^2 R = 0$. Let $R(r) = r^k$, then

$$r^2 R'' + rR' - n^2 R = (k(k-1) + k - n^2) r^k = 0,$$

which means $k = n, -n$. Thus, the general solution is

$$R(r) = c_1 r^n + c_2 r^{-n}.$$

As $r \to 0$, $r^{-n}$ does not converge. So, $c_2 = 0$ and the fundament solution corresponding to $\lambda = n^2$ is

$$u_n(r, \theta) = r^n (c_1 \cos(n\theta) + c_2 \sin(n\theta)).$$

Therefore, a solution to $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ is

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + d_n \sin n\theta).$$

Finally, the boundary condition $u(a, \theta) = f(\theta)$ yields

$$u(a, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + d_n \sin n\theta) = f(\theta).$$

Using the Fourier sine series of $f$, the coefficients $c_n$ and $d_n$ are determined by

$$\begin{aligned} c_n &= \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \\ d_n &= \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta. \end{aligned}$$

# Lecture 35. The Occurrence of Two-Point Boundary Value Problems (Sec 11.1)

Previously, we have seen the heat conduction equation $\alpha^2 u_{xx} = u_t$ with boundary conditions $u(0,t) = 0$ (or $u_x(0,t) = 0$) and $u(L,t) = 0$ (or $u_x(L,t) = 0$) and initial condition $u(x,0) = f(x)$. We used the method of separation of variables to deduce two ODEs

$$\begin{aligned} &X'' + \lambda X = 0, \qquad X(0) = X(L) = 0, \\ &T' + \alpha^2 \lambda T = 0. \end{aligned}$$

It turned out that the ODE for $X$ with the boundary conditions leads to eigenvalue problems. We have shown that for some $\lambda_n$, there exists nontrivial solutions for the boundary problem. Then, we solved the ODE for $T$ and used the superposition property to get the solution.

Our goal of this and the next lecture is to generalize the heat conduction problem. We consider the partial differential equations of the form

$$r(x)u_t = (p(x)u_x)_x - q(x)u$$

with boundary conditions

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0, \qquad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$. For example, the heat conduction problem is the case where $p(x) = 1 = r(x)$ and $q(x) = 0$.

Let $u(x,t) = X(x)T(t)$, then

$$\begin{aligned} r(x)X(x)T'(t) &= (p(x)X'(x))'T(t) - q(x)X(x)T(t) \\ \frac{T'(t)}{T(t)} &= \frac{(p(x)X'(x))'}{r(x)X(x)} - \frac{q(x)}{r(x)} = -\lambda. \end{aligned}$$

Thus, we have $T' + \lambda T = 0$

$$(p(x)X')' - q(x)X + \lambda r(x)X = 0.$$

The boundary conditions read

$$\alpha_1 X(0) + \alpha_2 X'(0) = 0, \qquad \beta_1 X(L) + \beta_2 X'(L) = 0$$

To solve the PDE $r(x)u_t = (p(x)u_x)_x - q(x)u$, it suffices to understand the eigenvalue problem $(p(x)X')' - q(x)X + \lambda r(x)X = 0$ with the boundary conditions. This is called Sturm–Liouville theory.

**Example 2.** Consider the case where $p(x) = r(x) = 1$, $q(x) = 0$, $\alpha_2 = 0$, $\alpha_1 = \beta_1 = \beta_2 = 1$, and $L = \pi$. That is, $X'' + \lambda X = 0$ with $X(0) = 0$ and $X(\pi) + X'(\pi) = 0$.

Suppose $\lambda = -\mu^2 < 0$, then

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x.$$

Note that $X(0) = c_1 = 0$ and

$$X(\pi) + X'(\pi) = c_2(\sinh \mu\pi + \mu \cosh \mu\pi) = 0.$$

If $c_2 \neq 0$, then $\mu$ satisfies $\sinh \mu\pi + \mu \cosh \mu\pi = 0$ and so

$$\mu = -\tanh \mu\pi.$$

Since $-\tanh \mu\pi < 0$ for $\mu > 0$, there is no such $\mu$. That is, there is no negative eigenvalue.

Suppose $\lambda = 0$, then $X(x) = c_1 + c_2 x$. By the boundary conditions, $c_1 = 0$ and $X(\pi) + X'(\pi) = c_2(\pi + 1) = 0$. Thus, $X(x) = 0$, which means that 0 is not an eigenvalue.

Suppose $\lambda = \mu^2 > 0$, then

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

Note that $X(0) = c_1 = 0$ and

$$X(\pi) + X'(\pi) = c_2(\sin \mu\pi + \mu \cos \mu\pi) = 0.$$

If $c_2 \neq 0$, then $\mu$ satisfies $\sin \mu\pi + \mu \cos \mu\pi = 0$ and so

$$\mu = -\tan \mu\pi.$$

For each $n \in \mathbb{N}$, there exists $\mu_n \in (n - \frac{1}{2}, n + \frac{1}{2})$ such that $\mu_n = -\tan \mu_n \pi$. For each eigenvalue $\lambda_n = \mu_n^2$, the corresponding eigenfunction is

$$\phi_n(x) = k_n \sin \sqrt{\lambda_n} x$$

for arbitrary constant $k_n$. Note that as $n \to \infty$ $\lambda_n = \mu_n^2 \approx (n - \frac{1}{2})^2$.

# References


[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley



Department of Mathematics, University of Illinois at Urbana-Champaign
*E-mail address:* daesungk@illinois.edu