Math 285 Lecture Note: Week 13

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Lecture 32. The Wave Equation: Vibrations of an Elastic String (Sec 10.7)

1 Nonzero initial velocity

We consider the wave equation

$$a^2 u_{xx} = u_{tt} \tag{1}$$

with the boundary condition

$$u(0,t) = 0, \qquad u(L,t) = 0 \tag{2}$$

and the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = g(x). \tag{3}$$

Using the same argument, we get

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right),$$

and $T'' + a^2\lambda T = 0$ with $T(0) = 0$. Since the general solution for T is

$$T_n(t) = k_1 \cos\left(\frac{n\pi a}{L}t\right) + k_2 \sin\left(\frac{n\pi a}{L}t\right),$$

the initial condition $T(0) = 0$ implies $k_1 = 0$. Thus,

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right)$$

is a solution to (1) with (2) and $u(x,0) = 0$. Since this boundary problem is homogeneous, the superposition property yields that

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right)$$

is also a solution. Finally, the initial condition $u_t(x,0) = g(x)$ yields

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi a}{L} \sin\left(\frac{n\pi}{L}x\right).$$

We use the Fourier sine series of g to determine C_n

$$D_n = \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right)\, dx.$$

2 General case

We are ready to find a solution $u(x,t)$ to (1) with boundary conditions (2) and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x). \tag{4}$$

To this end, we find two solutions $v(x,t)$ and $w(x,t)$ such that $v(x,t)$ is a solution with initial condition

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{5}$$

and $w(x,t)$ with (3). That is,

$$\begin{aligned} a^2 v_{xx} &= v_{tt}, \qquad v(0,t) = v(L,t) = 0, \qquad v(x,0) = f(x), \qquad v_t(x,0) = 0, \\ a^2 w_{xx} &= w_{tt}, \qquad w(0,t) = w(L,t) = 0, \qquad w(x,0) = 0, \qquad w_t(x,0) = g(x), \end{aligned}$$

Define $u(x,t) = v(x,t) + w(x,t)$, then

$$\begin{aligned} a^2 u_{xx} &= a^2(v_{xx} + w_{xx}) = a^2 v_{xx} + a^2 w_{xx} = v_{tt} + w_{tt} = u_{tt}, \\ u(0,t) &= v(0,t) + w(0,t) = 0, \\ u(L,t) &= v(L,t) + w(L,t) = 0, \\ u(x,0) &= v(x,0) + w(x,0) = f(x) + 0 = f(x), \\ u_t(x,0) &= v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x). \end{aligned}$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{n\pi a}{L}t\right) + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi a}{L}t\right)$$

with

$$\begin{aligned} C_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) \, dx, \\ D_n &= \frac{2}{n\pi a} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) \, dx. \end{aligned}$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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