Math 285 Lecture Note: Week 10

Daesung Kim

Lecture 23. Two-Point Boundary Value Problems, part 1 (Sec 10.1)

We consider $y'' + p(t)y' + q(t)y = g(t)$. Previously, the initial value problem refers the DE with the condition of the form

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

In this section, we will consider $y'' + p(x)y' + q(x)y = g(x)$ with

$$y(\alpha) = y_0, \qquad y(\beta) = y_1$$

for some $\alpha < \beta$. This is call a two-point boundary value problem. Our goal is to find solutions $y = \phi(x)$ that satisfies the DE in $x \in (\alpha, \beta)$ with the boundary condition.

Definition 1. A two-point boundary value problem is called *homogeneous* if $g(t) = y_0 = y_1 = 0$. Otherwise, we call it *nonhomogeneous*.

It is natural to ask if the Existence and Uniqueness theorem is available. In general, the answer is no. First, we consider nonhomogeneous case.

Example 2 (Nonhomogeneous with a unique solution)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\frac{\pi}{2}) = 2$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = 1$ and $C_2 = 2$. This shows that there exists a unique solution.

Example 3 (Nonhomogeneous with infinitely many solution)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = -1$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = 1$. Since there is no restriction on C_2, there are infinitely many solutions.

Example 4 (Nonhomogeneous with no solutions)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = 2$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

there are no C_1 and C_2 that satisfy the boundary condition. This shows that the solution does not exist.

If the boundary problem is homogeneous, we always have a trivial solution, $y(x) = 0$.

Example 5 (Homogeneous with infinitely many solutions)**.** Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\pi) = 0$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

we have $C_1 = 0$. Since there is no restriction on C_2, there are infinitely many solutions.

Example 6 (Homogeneous with a unique solution)**.** Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\frac{\pi}{2}) = 0$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = C_2 = 0$. This shows that there exists a unique solution $y(x) = 0$.

Lecture 24. Two-Point Boundary Value Problems, part 2 (Sec 10.1)

In this section, we focus on a homogenous boundary value problem $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ for some $L > 0$.

Recall that a boundary value problem $y'' + p(x)y' + q(x)y = g(x)$ with $y(\alpha) = y_0$ and $y(\beta) = y_1$ is called *homogeneous* if $g(x) = 0$ and $y_0 = y_1 = 0$. Otherwise, it is called *nonhomogeneous*.

If a boundary value problem is nonhomogeneous, it has (i) a unique solution, (ii) infinitely many solutions, or (iii) no solutions. If it is homogeneous, the problem always has a trivial solution $y = 0$. So, it has (i) a unique solution or (ii) infinitely many solutions.

Definition 7. Let $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ for $L > 0$. We call λ is an *eigenvalue* of the boundary value problem if it has nontrivial solutions. The solutions are called the corresponding *eigenfunctions*.

Our goal is to find all eigenvalues and eigenfunctions of $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ where $L > 0$.

Case 1: $\lambda > 0$.

For notational simplicity, let $\lambda = \mu^2$ for $\mu \in \mathbb{R}$. The general solution to $y'' + \mu^2 y = 0$ is

$$y(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

The boundary conditions yield $C_1 = 0$ and

$$y(L) = C_2 \sin \mu L = 0.$$

If $C_2 \neq 0$, then $\mu L = n\pi$ for $n \in \mathbb{N}$. Thus, if $\lambda \neq n^2(\pi/L)^2$ then the boundary value problem has a unique trivial solution. The eigenvalues are $\lambda = n^2(\pi/L)^2$ for all $n \in \mathbb{N}$ and the corresponding eigenfunctions are $C \sin(n\pi x/L)$.

Case 2: $\lambda < 0$.

Let $\lambda = -\mu^2$ for $\mu \in \mathbb{R}$. The general solution to $y'' - \mu^2 y = 0$ is

$$y(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

where $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. The boundary conditions yield $C_1 = 0$ and

$$y(L) = C_2 \sinh \mu L = 0.$$

Since $L \neq 0$, $C_2 = 0$. Thus, the equation does not have nontrivial solutions. There is no negative eigenvalue.

Case 3: $\lambda = 0$.

The general solution to $y'' = 0$ is $y(x) = C_1 x + C_2$. The boundary conditions yield $y(0) = C_2 = 0$ and $y(L) = C_1 L = 0$. Thus, the equation does not have nontrivial solutions and so 0 is not an eigenvalue.

Remark 8. In general, let $L[y]$ be a differential operator. For example, $L[y] = -y''$ or $L[y] = -x^2 y'' + 2xy'$. Suppose boundary conditions are given by $y(\alpha) = y_0$ (or $y'(\alpha) = y_0$) and $y(\beta) = y_1$ (or $y'(\beta) = y_1$). Then, λ is an eigenvalue of $L[y]$ with the boundary conditions if $L[y] = \lambda y$ with the boundary conditions has nontrivial solutions.

Lecture 25. Fourier Series, part 1 (Sec 10.2)

1 Periodic functions

Definition 9. A function f is periodic with period $T > 0$ if

(i) $x + T$ belongs to the domain of f if x does, and

(ii) $f(x + T) = f(x)$ for all x.

The smallest period $T > 0$ is called the fundamental period of f.

Example 10. It is easy to see that $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$ are periodic with the same period $2L/m$.

Proposition 11. *If f and g are periodic functions with common period T, then so is $c_1 f + c_2 g$ for any $c_1, c_2 \in \mathbb{R}$.*

2 Inner product and Orthogonality

Definition 12. For functions f and g on $[\alpha, \beta]$, we define the standard inner product of f and g by

$$(f, g) = \int_\alpha^\beta f(x)g(x)\,dx.$$

Remark 13. The inner product has the following properties:

(i) (Linearity) $(cf + g, h) = c(f, h) + (g, h)$;

(ii) (Symmetry) $(f, g) = (g, f)$;

(iii) (Positive-definite) $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$.

Indeed, if a relation $(\cdot, \cdot)$ satisfies these three assumptions, we call it an inner product. An elementary example of inner product is dot product.

Definition 14. We say that functions f and g are orthogonal on $[\alpha, \beta]$ if $(f, g) = 0$. We say that a set of functions are mutually orthogonal if any two functions in the set are orthogonal.

Example 15. One can see that

$$\begin{aligned}
&\left(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right) = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n, \end{cases} \\
&\left(\sin\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\right) = 0, \\
&\left(\cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\right) = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}
\end{aligned}$$

Thus, the set $\{\sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L} : m \in \mathbb{Z}\}$ is mutually orthogonal.

3 Fourier series

Suppose that a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right).$$

Assume that the infinite sum in the RHS converges for each $x \in [-L, L]$. Note that f is periodic with period $2L$. Our goal is to relate f with the coefficients a_m, b_m. To this end, we compute

$$(f, \cos\frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$(f, \sin\frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,$$

for each $n = 0, 1, 2, \cdots$. In fact, we have

$$\begin{aligned}(f, \cos\frac{n\pi x}{L}) &= \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\ &= a_n L\end{aligned}$$

by orthogonality for $n \in \mathbb{N}$. (We note that the above computation is not rigorous. To be precise, one needs to justify whether the infinite sum and integrals are interchangeable, and if the sum converges. This is beyond the scope of the course.) Similarly,

$$\begin{aligned}(f, \sin\frac{n\pi x}{L}) &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\ &= b_n L\end{aligned}$$

for $n \in \mathbb{N}$. If $n = 0$, then

$$(f, \cos\frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\,dx = a_0 L.$$

Therefore, we conclude that

$$a_n = \frac{1}{L}(f, \cos\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$b_n = \frac{1}{L}(f, \sin\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu