Math 285 Lecture Note: Week 1

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Lecture 1. Mathematical Models and Direction Fields (Sec 1.1)

A differential equation is an equation consisting of functions and their derivatives. Why are we interested in such equations? Suppose our goal is to understand phenomena like

(i) the motion of a falling object, fluids,

(ii) the flow of current in electric circuits,

(iii) the trend of populations, etc.

One way is to consider the rate of change. The rate of change can be described as a derivative. If we let $y(t)$ be a quantity that we want to analyze, then the rate of change of $y(t)$ is a derivative of y in time t, which denotes

$$\frac{dy}{dt} = y' = y_t.$$

Using some physics laws, observation, experimental results, or statistical inference, we can build a relation between the rate of change and other quantities like

$$\frac{dy}{dt} = F(t, y, \cdots).$$

This procedure is called a mathematical modeling. We end up obtaining a differential equation. This is a motivation of the study of differential equations.

Example 1. We are interested in understanding a falling object with air resistance. Using Newton's second law, one can build a mathematical model like

$$\frac{dv}{dt} = 5 - v.$$

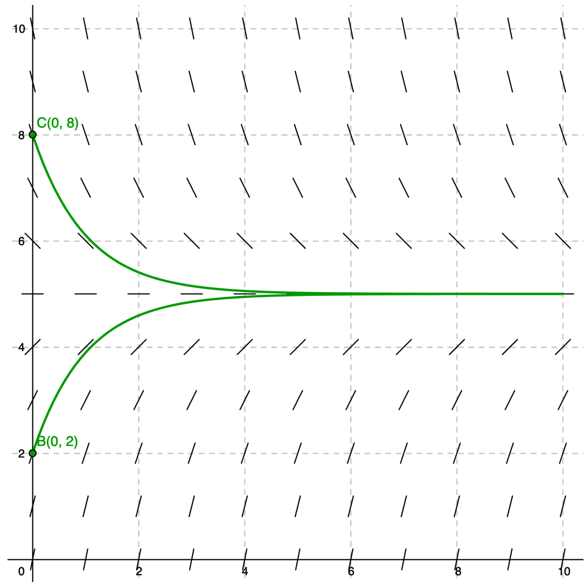
A following question is how to find the function v that satisfies the equation. Such v is called a solution. We first observe that

(i) the LHS represents the slope of v;

(ii) the RHS depends only on v, not on time t;

(iii) if $v > 5$, then $v' < 0$; if $v < 5$, then $v' > 0$; if $v = 5$, then $v' = 0$.

For further investigation, we draw a picture on $t - v$ graph.



This graph is called the direction field. The direction field suggests the following:

(i) There are many solutions.

(ii) If we specify one velocity (like at time 0), then there will be one solution.

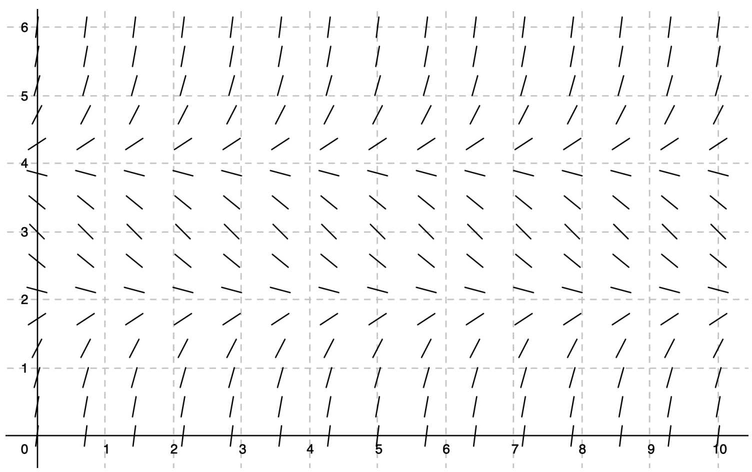
(iii) It is easy to see that $y(t) = 5$ is a solution. This is called an equilibrium solution.

(iv) The trend of velocity will be (dramatically) changed according to the choice of the initial value $y(0)$.

Example 2. Consider a differential equation

$$\frac{dp}{dt} = (p-2)(p-4).$$

The direction field is



Lecture 2. Solutions of Some Differential Equations (Sec 1.2)

In this section, we study differential equations of the form

$$\frac{dy}{dt} = F(t, y).$$

In particular, we focus on easy cases where the function F depends only on t or on y.

Example 3. Consider a differential equation

$$\frac{dy}{dt} = t^2.$$

If the RHS is a function of t, then finding a solution $y = y(t)$ is just a matter of integration. Indeed,

$$f(t) = \frac{1}{3}t^3 + C$$

for some constant C.

Now, we focus on the case where $F(t, y)$ is a function of y.

Example 4. Consider a differential equation

$$\frac{dv}{dt} = 5 - v.$$

Then, we have

$$\frac{1}{5-v}\frac{dv}{dt} = 1$$

The idea is to exploit the chain rule. Recall that

$$\frac{d}{dt}G(v(t)) = G'(v)\frac{dv}{dt}.$$

It suffices to find G satisfying

$$G'(v) = \frac{1}{5-v}.$$

Once finding such function G, the equation can be written as

$$\frac{1}{5-v}\frac{dv}{dt} = G'(v)\frac{dv}{dt} = \frac{d}{dt}G(v(t)) = 1.$$

Now the RHS does not depend on v. By integrating of both sides, we get

$$G(v(t)) = t + C.$$

How do we find such G? Taking integration of both sides, we get

$$G(v) = \int \frac{1}{5-v}\, dv = -\ln|5-v| + C.$$

We can just pick one function G. So, let $G(v) = -\ln|v-5|$, then

$$\begin{aligned} -\ln|v-5| &= t + C \\ |v-5| &= e^{-t+C} \\ v &= 5 + Ce^{-t} \end{aligned}$$

where C is an arbitrary constant. (Note that the constants C may be different from line to line.)

The key idea is to replace the original equation to easier one. This principle is one of the main theme in this course.

Example 5. Consider a differential equation

$$\frac{dy}{dt} = 1 + y^2.$$

Similarly, we let $G'(y) = \frac{1}{1+y^2}$ and solve this auxiliary differential equation. By taking integration, we have

$$G(y) = \arctan(y) = t + C$$

and so $y(t) = \tan(t + C)$.

Example 6. Consider a differential equation

$$\frac{dp}{dt} = (p-2)(p-4).$$

Similarly,

$$\frac{1}{(p-2)(p-4)}\frac{dp}{dt} = 1.$$

Let

$$G'(p) = \frac{1}{2}\left(\frac{1}{p-4} - \frac{1}{p-2}\right),$$

then

$$\begin{aligned} G(p) &= \frac{1}{2}\int \left(\frac{1}{p-4} - \frac{1}{p-2}\right)\, dp \\ &= \frac{1}{2}\left(\ln|p-4| - \ln|p-2|\right) \\ &= \frac{1}{2}\ln\left|\frac{p-4}{p-2}\right| \\ &= t + C. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{p-4}{p-2} &= 1 - \frac{2}{p-2} = Ce^{2t}, \\ \frac{2}{p-2} &= 1 + Ce^{2t}, \\ p(t) &= 2 + \frac{2}{1 + Ce^{2t}}. \end{aligned}$$

Lecture 3. Classification of Differential Equations (Sec 1.3)

There are several types of differential equations. Suppose that we consider a differential equation with unknown function f. This function f can be a function of one variable (which represents time, for example) or several variables (describing space and time). Let

$$f = f(t, x, \cdots).$$

Definition 7. An *ordinary differential equation (ODE)* is an equation with $f = f(t)$ and the derivatives in one variable. A *partial differential equation (PDE)* is an equation with $f = f(t, x, \cdots)$ and the derivatives in several variables.

Example 8.

$$\begin{aligned}
\frac{d^2 f}{dx^2} + f = 0 \qquad & \text{(Harmonic oscillator)} \\
\frac{d^2 f}{dx^2} + \sin(f) = 0 \qquad & \text{(Motion of pendulum)} \\
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad & \text{(Laplace equation)} \\
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \qquad & \text{(Heat equation)} \\
\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} \qquad & \text{(Wave equation)}
\end{aligned}$$

Notation 9. We use the notations

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \qquad \frac{\partial^2 f}{\partial x \partial t} = f_{xt}, \cdots$$

Definition 10. A *system of differential equations* is a collection of differential equations with several unknown functions. For example,

$$\begin{cases} \dfrac{df}{dx} = 3f(x) - g(x), \\ \dfrac{dg}{dx} = 2f(x) + g(x). \end{cases}$$

Definition 11. The *order* of a differential equation is the order of the highest derivative that appears in the equation. For example, the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

is of order 2.

Another way to categorize differential equations is linearity. In general, a differential equation can be written as

$$F(t, x, \cdots, f, f', f'', \cdots) = 0$$

where $t, x, \cdots$ are variables and f, F are functions.

Definition 12. We consider an ODE given by

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

where y is a function of t and $y^{(n)}$ denotes the n-th derivative. We say the ODE is *linear* if F is linear in $y, y', \cdots, y^{(n)}$, that is,

$$F(t, y, y', \cdots, y^{(n)}) = a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y + g(t) = 0.$$

If not, it is called *nonlinear*.

Example 13. Note that

$$y'' + 2y' - 3y = \tan(t)$$

is a second-order linear differential equation and

$$y'' + \sin(y) = 0, \qquad yy'' = y^2$$

are nonlinear equations.

Definition 14. A *solution* of the ODE given by

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

on the interval (α, β) is a function $\phi(t)$ such that

$$F(t, \phi, \phi', \cdots, \phi^{(n)}) = 0$$

for every $t \in (\alpha, \beta)$.

Example 15. Note that $y = \phi(t) = e^t$ is a solution to the second order linear ODE

$$y'' - y = 0$$

for all $t \in \mathbb{R}$. To verify this, we substitute y with e^t, i.e.,

$$\phi''(t) - \phi(t) = e^t - e^t = 0.$$

Note also that e^t is not the only solution. $\psi(t) = e^{-t}$ is also a solution. Furthermore, any linear combination of $\phi(t)$ and $\psi(t)$ is also a solution. That is,

$$f(t) = a\phi(t) + b\psi(t)$$

is a solution for $a, b \in \mathbb{R}$. A particular example of this linear combination is $\sinh(t)$ and $\cosh(t)$.

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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