Solutions to Midterm 3

1. Suppose
$$0 < \lambda_1 < \lambda_2 < \cdots$$
are the eigenvalues of $y'' + \lambda y = 0$ with $y(0) = y(\pi) = 0$ in the interval $[0, \pi]$. Let $\varphi_k(x)$ be the eigenfunction associated to λ_k. What is the number of x's in the interval $[0, \pi]$ such that $\varphi_k(x) = 0$?

> **Solution:** The k-th eigenvalue and eigenvector are $\lambda_k = k^2$ and $\varphi_k(x) = \sin(kx)$. Since $\sin(kx) = 0$ for $x = \frac{i}{k}\pi$ for $i = 0, 1, 2, \cdots, k$, the answer is $k + 1$.

2. Let f and g be functions defined on $\mathbb{R}$. Which one of the followings is NOT correct?

> **Solution:**
>
> (i) If $f(x) = e^{x^2}$ and $g(x) = 3\cos(x)$, then $\int_{-1}^{1} f(x)g(x)\,dx = 0$: False because f and g are even and strictly positive on $[0, 1]$.
>
> (ii) If f is odd, then $f(0) = 0$: True. Since $f(x) = -f(-x)$ for all x, in particular, we have $f(0) = -f(-0) = -f(0)$, which implies $f(0) = 0$.
>
> (iii) If g is even, then x and $g(x)$ are orthogonal on $[-6, 6]$ with respect to the inner product $\langle \varphi, \psi \rangle = \int_{-6}^{6} \varphi(x)\psi(x)\,dx$: True. Since x is odd and $g(x)$ is even, $xg(x)$ is odd. Thus, the integral of $xg(x)$ over $[-6, 6]$ is zero.
>
> (iv) If f is even and odd, then $f(x) = 0$ for all $x \in \mathbb{R}$: True. Since f is even and odd, we have $f(x) = f(-x) = -f(-x)$ for all x. In particular, we have $f(x) = -f(x)$ for all x, which yields $f(x) = 0$ for all x.
>
> (v) If g is periodic with period 16, then a function $h(x) = g(4x - 3)$ is also periodic with period 4: True. We have $h(x + 4) = g(4(x + 4) - 3) = g((4x - 3) + 16) = g(4x - 3) = h(x)$.

3. Let $f(x)$ be a function on $[0, 5]$ given by $f(x) = 4x$. Find the Fourier sine series for $f(x)$ of period 10.

Solution: The sine series $S(x)$ is

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} C_n \sin\left(\frac{m\pi}{5}x\right) \\ C_n &= \frac{2}{L}\int_0^5 f(x)\sin\left(\frac{m\pi}{5}x\right) \\ &= \frac{8}{5}\int_0^5 x\sin\left(\frac{m\pi}{5}x\right). \end{aligned}$$

4. Consider the function $f(x)$ defined on $\mathbb{R}$ such that $f(x) = f(x+8)$ and

$$f(x) = \begin{cases} x-2, & -4 \le x < 0, \\ 1, & x = 0, \\ x^2+10, & 0 < x < 4. \end{cases}$$

Let $S(x)$ be the Fourier series of $f(x)$. What is $S(12)$?

Solution: By the Fourier convergence theorem and the periodicity, we have

$$\begin{aligned} S(12) &= \frac{1}{2}(f(12-) + f(12+)) \\ &= \frac{1}{2}(f(4-) + f((-4)+)) \\ &= \frac{1}{2}(\lim_{x\to 4-}(x^2+10) + \lim_{x\to(-4)+}(x-2)) \\ &= \frac{1}{2}(26-6) = 10. \end{aligned}$$

5. What is the steady state solution $v(x)$ for the following heat conduction equation?

$$\begin{cases} u_{xx} = 4u_t, & \text{for } 0 < x < 7, \quad t > 0, \\ u(0,t) = 8, & \text{for } t \ge 0, \\ u(7,t) = 57, & \text{for } t \ge 0, \\ u(x,0) = x^2+8, & \text{for } 0 \le x \le 7 \end{cases}$$

Solution: The steady state solution $v(x)$ is a solution to $v'' = 0$ with $v(0) = 8$ and $v(7) = 57$. Thus, $v(x) = ax + b$, $b = 8$, and $v(7) = 7a + 8 = 57$, that is $a = 7$.

6. Consider the following equation for $y(x)$:

$$y'' + \lambda\, y = 0$$

and the following eigenfunctions: $y_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$, $\quad n \geq 1$ Which boundary conditions would give the above eigenfunctions?

A. $y(0) = 0; \quad y'(L) = 0$

B. None of these

C. $y'(0) = 0; \quad y'(2L) = 0$

D. $y(0) = 0; \quad y(L) = 0$

E. $y(0) = 0; \quad y'(2L) = 0$

Solution: The boundary value problem $y'' + \lambda\, y = 0$ with $y(0) = 0; \quad y'(L) = 0$ has no solutions for $\lambda < 0$ and for $\lambda = 0$. For $\lambda > 0$ we have that the general solution to $y'' + \lambda\, y = 0$ is $y = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$ and plugging $x = 0$ gives $0 = y(0) = c_1$ and using this and plugging in $x = L$

$$0 = y'(L) = c_2 \sqrt{\lambda} \cos(\sqrt{\lambda} L)$$

which is possible if and only if $\sqrt{\lambda} L = \frac{2n-1}{2}\pi$

7. Consider the following boundary value problem for $y(x)$: $y'' + y = a\, x^2$ $y(0) = 0$ $y(L) = \beta\, \pi^2 - \gamma$ Here $L = [\pi, \pi, \pi/2]$ $\beta = [c, 2c, c]$ $\gamma = [4c, 8c, 4c]$

How many solutions does this problem have?

A. Infinite solutions (**correct if** $L = \pi, \beta = c, \gamma = 4c$)

B. No solution (**correct if** $L = \pi, \beta = 2c, \gamma = 8c$)

C. One solution(**correct if** $L = \frac{\pi}{2}, \beta = c, \gamma = 4c$)

D. Two solutions

E. Cannot be determined

Solution: The general solution to $y'' + y = a\, x^2$ is $y = c_1 \cos x + c_2 \sin x + ax^2 - 2a$ Imposing $y(0) = 0$ we have $0 = c_1 - 2a$ or $c_1 = 2a$ and imposing $0 = y(L)$ we have $0 = 2a \cos x + c_2 \sin x + 2aL^{-}2a$ plugging in the various values for L β and γ gives the result(s).

8. Find the solution $u(x,t)$ where $0 < x < L$ and $t > 0$ of the diffusion equation¡/p¿ $u_t = \kappa\, u_{xx}$ $u\prime(0,t) = 0;$, $\quad u\prime(L,t) = 0;$ $\quad u(x,0) = a$

   A. $u(x,t) = a$

   B. $u(x,t) = \sum_{m=1}^{\infty} a\frac{(n\pi)^2}{L}\sin(\frac{m\pi}{Lx})e^{-\frac{n^2\pi^2}{L^2}t}$

   C. $u(x,t) = \sum_{m=1}^{\infty} a\,\frac{(n\pi)^2}{L}\cos(\frac{m\pi}{Lx})e^{-\frac{n^2\pi^2}{L^2}t}$

   D. $u(x,t) = \sum_{m=1}^{\infty} a\,\sin(\frac{m\pi}{Lx})e^{-\frac{n^2\pi^2}{L^2 t}}$

   E. $u(x,t) = \sum_{m=1}^{\infty} a\,\sin(\frac{m\pi}{Lx})e^{-\frac{n^2\pi^2}{L^2 t}}$

Solution:

The solution is $u(x,t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_n\frac{(n\pi)^2}{L}\cos(\frac{m\pi}{Lx})e^{-\frac{n^2\pi^2}{L^2}t}$ where a_n = sine Fourier coefficients of a or

$$a_0 = \frac{2}{L}\int_0^L a dx = 2a \qquad a_n = \frac{2}{L}\int_0^L a\,\cos(\frac{n\pi}{Lx})dx = 0$$

thus

$$u(x,t) = a$$

9. Let $\tilde{f}(x)$ be the even extension to the interval $-L < x < L$ of the function $f(x) = a\,x$ defined on $0 < x < L$.

   If the Fourier series of $\tilde{f}(x)$ in the interval $-L < x < L$ is

$$S(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})),$$

   find A_n and B_n for $n \geq 1$. You don't need to calculate A_0.

Solution:

Since the even extension is $\tilde{f}(x) = a|x|$ a straightforward calculation (done in the lectures) gives

$$A_n = \frac{1}{L}\int_{-L}^{L} a|x|\cos(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L a|x|\cos(\frac{n\pi x}{L})dx = \frac{2aL(\cos(n\pi) - 1)}{(n^2\pi^2)}$$

and

$$B_n = \frac{1}{L}\int_{-L}^{L} a|x|\sin(\frac{n\pi x}{L})dx = 0$$

where $int^L_{-L}a|x|\sin(\frac{n\pi x}{L})dx = 0$ because $int^L_{-L}a|x|\sin(\frac{n\pi x}{L})$ = eve × odd = odd

10. Using separation of variables solve the following diffusion equation problem for $u(x,t)$ where $0 < x < L$ and $t > 0$ $u_t = \kappa\, u_{xx}$ $u'(0,t) = 0;\qquad u'(L,t) = 0$ $u(x,0) = a\,x$

Assume that the solution has the form $u(x,t) = \sum_{n=1}^{\infty} C_n\, T_n(t)\, X_n(x)$ Calculate C_n, $T_n(t)$, and $X_n(x)$ for $n \geq 1$ (i.e., ignore C_0, T_0, X_0)

Solution:

$$C_n = \frac{2aL(\cos(n\pi) - 1)}{(n^2\pi^2)},\ X_n = \cos(\frac{n\pi x}{L}),\ T_n = e^{-\frac{\kappa n^2 \pi^2 t}{L^2}}$$

where the computation for C_n is the same as the one for B_n in the previous problem