Solutions to Practice problems for midterm 3

1. Consider the function defined as

$$f(x) = \begin{cases} x + a, & x \le 0, \\ b - x, & x > 0. \end{cases}$$

To which value will the Fourier Series expansion of $f(x)$ converge at $x = 0$?

Solution: It converges to the average of the left and right limit at $x = 0$, that is $\frac{1}{2}(f(0-) + f(0+)) = \frac{a+b}{2}$.

2. Consider the function $f(x) = a$ for $0 < x < L$. Which one is the Fourier cosine series expansion of $f(x)$?

   A. a
   B. 0
   C. $\sum_{n=1}^{\infty} \frac{a}{n\pi} \cos\left(\frac{n\pi x}{L}\right)$
   D. $\sum_{n=1}^{\infty} \frac{a}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right)$
   E. $\sum_{n=1}^{\infty} \frac{-2a}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$
   F. $\sum_{n=1}^{\infty} \frac{-2a[\cos(n\pi)-1]}{n^2\pi^2} \cos\left(\frac{n\pi x}{L}\right)$

Solution: The Fourier cosine series $S(x)$ is

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L}x), \qquad a_n = \frac{2a}{L}\int_0^L \cos(\frac{n\pi}{L}x)\,dx \text{ for } n = 0, 1, 2, \cdots.$$

Since $a_n = 0$ for all $n = 1, 2, \cdots$ and $a_0 = 2a$, we have $S(x) = a$ for all x.

3. Consider the following diffusion equation problem for $u(x,t)$:

$$\begin{cases} u_t = \gamma\, u_{xx} \\ u(0,t) = 0; \qquad u(L,t) = 0 \\ u(x,0) = \gamma\, x^2 \end{cases}$$

If you were to use separation of variables and write $u(x,t) = X(x)\,T(t)$, which would be the correct boundary conditions for $X(x)$?

   A. $X(0) = 0; \qquad X(2L) = 0$
   B. $X'(0) = 0; \qquad X'(L) = 0$
   C. $X'(0) = 0; \qquad X'(L) = 0$

D. $X(0) = 0; \qquad X(\gamma) = 0$

E. $X(0) = 0; \qquad X(L) = 0$

F. $X'(0) = 0; \qquad X'(\gamma) = 0$

G. None of these

Solution: Since

$$u_t = X(x)T'(t) = \gamma X''(x)T(t) = \gamma u_{xx},$$

the separation of variable yields $\frac{X''(x)}{X(x)} = \frac{1}{\gamma}\frac{T'(t)}{T(t)} = -\lambda$. From the boundary conditions, we have $X(0)T(t) = 0 = X(L)T(t)$. If $X(0) \neq 0$ or $X(L) \neq 0$, then $T(t) = 0$ so that $u(x,t) = 0$. Thus, ther correct boundary conditions for X are $X(0) = X(L) = 0$.

4. Consider the following boundary value problem for $y(x)$

$$y'' + y = a\,x$$

with $y(0) = 0$ and $y(L) = \beta\pi$. Here L is either $L = \pi$ or $L = \frac{\pi}{2}$ and β is either $\beta = a$ or $\beta = 2a$. How many solutions does this problem have?

A. Two solutions

B. Cannot be determined

C. No solution

D. Infinite solutions

E. One solution

.

Solution: The general solution is $y(x) = ax + C_1\cos(x) + C_2\sin(x)$. Since $y(0) = 0$, we have $C_1 = 0$.

Suppose $L = \pi$. Then, $y(\pi) = a\pi = \beta\pi$. If $\beta = a$, then there are infinitely many solutions. If $\beta = 2a$, then there is no solution.

Suppose $L = \frac{\pi}{2}$. Then, $y(\frac{\pi}{2}) = \frac{\pi}{2}a + C_2 = \beta\pi$. So, there is one solution.

5. Consider the following equation for $y(x)$:

$$y'' + \lambda\,y = 0$$

and the following eigenfunctions $y_0(x) = 1$ and

$$y_n(x) = \cos\left(\frac{n\pi x}{L}\right).$$

Which boundary conditions would give the above eigenfunctions?

A. $y'(0) = 0; \qquad y'(2L) = 0$
B. $y(0) = 0; \qquad y(L) = 0$
C. $y(0) = 0; \qquad y(2L) = 0$
D. $y'(0) = 0; \qquad y'(L) = 0$
E. None of these

Solution: One can see that

$$y_n'(x) = -\frac{n\pi}{L}\sin(\frac{n\pi x}{L})$$

and $y_n'(0) = y_n'(L) = 0$.

6. Consider the following eigenvalue problem for $y(x)$:

$$y'' + \lambda\, y = 0$$

with $y(0) = 0$ and $y'(L) = 0$. Which are the correct eigenvalues and corresponding eigenfunctions?

A. $\lambda_n = \frac{(2n-1)^2\pi^2}{4L^2}; \qquad y_n(x) = \sin\left[\frac{(2n-1)\pi x}{2L}\right]$
B. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \cos\left[\frac{n\pi x}{L}\right]$
C. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \sin\left[\frac{n\pi x}{L}\right]$
D. $\lambda_n = \frac{(2n-1)^2\pi^2}{2L}; \qquad y_n(x) = \cos\left[\frac{(2n-1)\pi x}{L}\right]$
E. None of these

Solution: One can see that λ should be positive. Let $\lambda = \mu^2$. The general solution is $y(x) = C_1\cos(\mu x) + C_2\sin(\mu x)$. Then, $y(0) = C_1 = 0$ and

$$y'(L) = C_2\mu\cos(\mu L) = 0.$$

Since we are looking for nontrivial solutions and $C_1 = 0$, C_2 should be nonzero. Thus,

$$\mu L = \frac{(2n-1)\pi}{2}$$

for $n = 1, 2, \cdots$.

7. Consider the function $f(x) = a$ for $0 < x < L$. Which one is the Fourier sine series expansion of $f(x)$?

A. 0

B. $\sum_{n=1}^{\infty} \frac{a}{n\pi} \cos\left(\frac{n\pi x}{L}\right)$

C. $\sum_{n=1}^{\infty} \frac{-2a[\cos(n\pi)-1]}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$

D. $\sum_{n=1}^{\infty} \frac{a[\cos(n\pi)-1]}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right)$

E. $\sum_{n=1}^{\infty} \frac{a}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$

F. $\sum_{n=1}^{\infty} \frac{-2a}{n^2\pi^2} \sin\left(\frac{n\pi x}{L}\right)$

G. None of these

Solution: The Fourier sine series $S(x)$ is

$$S(x) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi}{L}x)$$

with

$$\begin{aligned} a_n &= \frac{2a}{L}\int_0^L \sin(\frac{n\pi}{L}x)\,dx \\ &= \frac{2a}{L}\left(-\frac{L}{n\pi}[\cos(\frac{n\pi}{L}x)]_0^L\right)\int_0^L \sin(\frac{n\pi}{L}x)\,dx \\ &= -\frac{2a}{n\pi}\left(\cos(n\pi)-1\right). \end{aligned}$$

8. Determine for which value of a the following boundary value problem for $y(x)$ has solutions.

$$y'' + y = 0, \qquad y(0) = \alpha, \qquad y(\pi) = a.$$

Solution: The general solution is $y(x) = C_1\cos(x) + C_2\sin(x)$. By the boundary conditions, we have $C_1 = \alpha$ and $-C_1 = a$. If $a = -\alpha$, then there are infinitely many solutions. Otherwise, there is no solution.

9. Consider the diffusion equation problem for $u(x,t)$ below:

$$\begin{cases} u_t = \gamma\, u_{xx} \\ u(0,t) = a; \qquad u(L,t) = b \\ u(x,0) = \gamma\, x \end{cases}$$

What is the steady state solution of this problem?

Solution: The steady state solution $v(x)$ is a solution to $v'' = 0$ with $v(0) = a$ and $v(L) = b$. Since the general solution is $v(x) = C_1 x + C_2$, we get $v(x) = a + \frac{b-a}{L} x$.