Solutions to Midterm 2

1. Which of the following are linearly dependent? Select all that apply.

Solution:

(i) $x^2, x+1, x^2+2x+2$: Since $x^2+2(x+1)-(x^2+2x+2)=0$, they are linearly dependent.

(ii) $e^x, 3e^x-2e^{-x}, 7e^x+5e^{-x}$: Since $-29e^x+5(3e^x-2e^{-x})+2(7e^x+5e^{-x})=0$, they are linearly dependent.

(iii) $\cos(x), \sin(x), -\cos(x+2)$: Since $\cos(x+2)=\cos(2)\cos(x)-\sin(2)\sin(x)$, they are linearly dependent.

(iv) $\ln(2x), 1, \ln(3x)$: Since $\ln(2x)-\ln(3x)=\ln((2x)/(3x))=\ln(\frac{2}{3})$, they are linearly dependent.

(v) $\sin(x), \sin(2x), \sin(3x)$: Let $a\sin(x)+b\sin(2x)+c\sin(3x)=0$. Since this holds for all x, put $x=\frac{\pi}{2}$, then $a-c=0$. Put $x=\frac{\pi}{3}$, then $a+b=0$. Taking derivative on the LHS and putting $x=\frac{\pi}{2}$, then $b=0$. Therefore, we get $a=b=c=0$. They are linearly independent.

(vi) $\cos(x), -\tan(x), \cos(x)+2$: Let $a\cos(x)+b(-\tan(x))+c(\cos(x)+2)=0$. Putting $x=0$, we have $a+3c=0$. Taking derivative and putting $x=0$, we have $b=0$. Thus, the equation can be written as $c(\cos(x)-1)=0$. If $x=\frac{\pi}{2}$, then $c=0$. Thus $a=b=c=0$ and so they are linearly independent.

(vii) $1, x, 2x+x^2$: Let $a+bx+c(2x+x^2)=0$. If $x=0$, then $a=0$. Taking derivatives on both sides, we get $b+c(2+2x)=0$ and $2c=0$. So we have $a=b=c=0$, which implies that they are linearly independent.

(viii) $xe^{2x}, -4x, e^{2x}$: Let $axe^{2x}+b(-4x)+c(e^{2x})=0$. Putting $x=0$, we have $c=0$. Taking derivative and putting $x=0$, $a-4b=0$. Repeating this, we end up with $a=b=c=0$, which implies that they are linearly independent.

(ix) $1+x^2+x^4, x^2+x^4, x^4$: The same method applies. They are linearly independent.

(x) $x, x\ln(2x), (x+1)\ln(2x)$: Let $ax+bx\ln(2x)+c(x+1)\ln(2x)=0$. Put $x=\frac{1}{2}$, then $a=0$. Taking derivative, we get

$$b(\ln(2x)+1)+c(\ln(2x)+1+\frac{1}{x})=0.$$

Let $x=\frac{1}{2}$, then we get $b+3c=0$. Repeating this, we obtain $b-c=0$. Thus, $a=b=c=0$ and so they are linearly independent.

(xi) $(x+1)^2, (x+2)^2, (x+3)^2$: Let $a(x+1)^2 + b(x+2)^2 + c(x+3)^2 = 0$. Let $x = -1, -2, -3$, then $b + 4c = a + c = 4a + b = 0$. Thus, $a = b = c = 0$ and so they are linearly independent.

(xii) e^x, e^{2x}, e^{3x}: Let $ae^x + be^{2x} + ce^{3x} = 0$ repeat the previous procedure: taking derivative and putting $x = 0$. So we obtain $a+b+c = a+2b+3c = a+4b+9c = 0$. Thus, $a = b = c = 0$ and so they are linearly independent.

2. Identify the correct form of a particular solution to

$$y'' + 6y' - 7y = 3x(e^{7x} + 3e^x) + e^x.$$

Solution: First, the fundamental solutions to the homogeneous equation are e^x, e^{-7x}. The non-homogeneous terms are $(9x+1)e^x$ and $3xe^{7x}$. For the first term, since e^x and xe^x are solutions to the homogeneous equation, a particular solution is of the form

$$Y_1(t) = x^2(B_0 + B_1 x)e^x.$$

For the second term, e^{-7x} is not a solution to the homogeneous equation so that

$$Y_2(t) = (A_0 + A_1 x)e^{-7x}.$$

3. Which of the following are correct? Select all that apply.

Solution:

(i) $u'' + 4u = 5\cos(2t)$ is in resonance.: This is true because $2 = \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}}$.

(ii) $u'' + 2u' + 2u = 0$ is underdampted.: This is true because the equation has complex roots.

(iii) If $u(t) = -\cos(t) + \sqrt{3}\sin(t) = R\cos(t - \delta)$, then $R = 2$.: This is true because $R = \sqrt{A^2 + B^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$.

(iv) The natural frequency of $3u'' + 9u = 0$ is $\sqrt{3}$.: This is true because $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = \sqrt{3}$.

(v) $u'' + 4u' + 4u = 0$ is overdampted.: Since the equation has double roots, it is critically damped.

(vi) $u'' + 3u' + 2u = 0$ is critically dampted.: Since the equation has distinct two real roots, it is overdamped.

(vii) $\sin(3x) + \cos(3x) = \sqrt{2}\cos(3x + \frac{\pi}{4})$: This is false because the RHS should be $\sqrt{2}\cos(3x - \frac{\pi}{4})$.

(viii) $\sin(3x) + \sin(5x) = \sin(4x)\cos(x)$: This is false because the RHS should be $2\sin(4x)\cos(x)$.

(ix) The period of $4u'' + u = 0$ is π.: This is false because the period is $\frac{2\pi}{\omega_0}$ and $\omega = \sqrt{\frac{k}{m}} = \frac{1}{2}$.

(x) The amplitude of $u(t) = 2\cos(2t) - 4\sin(2t)$ is $2^2 + (-4)^2 = 20$.: The amplitude is $R = \sqrt{2^2 + (-4)^2} = \sqrt{20}$.

4. Consider a differential equation

$$y'' - 18y' + 81y = 0.$$

Which of the following are solutions to the equation? Select all that apply.

(i) 0

(ii) e^{-9t}

(iii) e^{9t}

(iv) t^2e^{9t}

(v) 1

(vi) $te^{9t} + 3$

(vii) $(2t + 3)e^{9t}$

Solution: Since the characteristic equation $r^2 - 18r + 81 = 0$ has double root $r = 9$, the general solution is

$$y(t) = e^{9t}(C_1 + C_2t).$$

Thus, 0, e^{9t}, and $(2t + 3)e^{9t}$ are solutions.

5. If $t = 5\ln x$, then

$$\frac{d^2y}{dt^2} = F(x)\frac{dy}{dx} + G(x)\frac{d^2y}{dx^2}.$$

Find $F(x)$ and $G(x)$.

Solution: Since $x = e^{t/5}$, we have $\frac{dx}{dt} = \frac{1}{5}e^{\frac{t}{5}} = \frac{1}{5}x$. By chain rule and product rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{5}x\frac{dy}{dx}$$

and

$$\begin{aligned}\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{1}{5}x\frac{dy}{dx}\right)\\ &= \frac{1}{5}\frac{dx}{dt}\frac{dy}{dx} + \frac{1}{5}x\frac{d^2y}{dx^2}\frac{dx}{dt}\\ &= \frac{1}{25}x\frac{dy}{dx} + \frac{1}{25}x^2\frac{d^2y}{dx^2}.\end{aligned}$$

6. Consider an oscillator governed by

$$y'' + 10y' + ky = 0.$$

Let $D = 10^2 - 4k$. If $D > 0$, then it is overdamped. If $D = 0$, then it is critically damped. If $D < 0$, then it is underdamped.

7. Which of the following statements are true?

(i) If f and g are differentiable functions defined on $(-1, 1)$ and the Wronskian $W[f, g] = 0$ on $(-1, 1)$, then f and g are linearly dependent.

(ii) If $f(t) = e^{rt}$ and $g(t) = e^{st}$, then $W[f, g](t) \neq 0$ for all real numbers t if and only if $r \neq s$.

(iii) $\{\cos(x), \cos(2x), \cos(3x)\}$ is linearly independent.

Solution:

(i) False: if $f(x) = x|x|$ and $g(x) = x^2$, then the Wronskian is zero but they are linearly independent.

(ii) Since $W[f, g](t) = (s - r)e^{(r+s)t}$, $W[f, g](t) \neq 0$ for all real numbers t if and only if $r \neq s$.

(iii) Let $a\cos(x)+b\cos(2x)+c\cos(3x) = 0$. Putting $x = 0, \frac{\pi}{2}, \frac{\pi}{6}$, we get $a+b+c = 0$, $b = 0$, and $\sqrt{3}a+b = 0$. Solving the system of equations, we have $a = b = c = 0$, which implies that $\{\cos(x), \cos(2x), \cos(3x)\}$ is linearly independent.

8. Consider the following initial value problem for $y(x)$:

$$(x-9)y''' + \frac{10}{(x-15)}y'' - \frac{14}{(x-8)}y' + (x+15)y = \frac{(x-9)}{(x-14)}$$

with initial conditions $y(10) = -2, y'(10) = 5, y''(10) = 4$

On which interval are we guaranteed a unique solution?

Solution: Dividing by $(x-9)$ throughout the equation becomes

$$y''' + \frac{10}{(x-15)(x-9)}y'' - \frac{14}{(x-8)(x-9)}y' + \frac{(x+15)}{(x-9)}y = \frac{(x-9)}{(x-14)(x-9)}$$

which is of the form $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = g(x)$. The theorem of existence and uniqueness for linear equations guarantees a unique solution as long as $p_1(x)$, $p_2(x)\, p_3(x)$ and $g(x)$ are continuous. This happens for $x \neq -5, 8, 9, 14$ and since we must take **connected** intervals, the connected interval in which the initial condition is imposed, namely $x_0 = 10$, is the interval $(9, 14)$.

9. Consider the following oscillator equation for $x(t)$ and the given initial conditions:¡/p¿ $x'' + 1600\,x = 5\,\cos(41t)$ with $x(0) = 0; \qquad x'(0) = 0$ What is the long term behavior of the solution?

Solution: The characteristic polynomial is $r^2 + 1600$ whose roots are $r = \pm 40i$, therefore $\omega_0 = 40 \neq \omega = 41$ The general solution is

$$y(t) = A\cos(40t - \delta) + A\cos(41t)$$

For some value of A (find it!) and therefore **The solution will oscillate forever**.

10. Consider $y_1 = 3\,|x|x$ and $y_2 = 3x^2$ defined in $(-1, 1)$. Are they lin. indep.? What is $W(y_1, y_2)$?

Solution: $y_1 = 3\,|x|x$ thus $y_1 = -3x^2$ for $x < 0$ and $3x^2$ for $x > 0$. So $y_1' = 6|x|$ (which works at $x = 0$ as well) and

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 18|x|x^2 - 18|x|x^2 = 0$$

But they are linearly independent. In fact, if $0 = ay_1 + by_2 = 3a\,|x|x - 3bx^2$ and this last quantity equals $3(b-a)x^2$ if $x < 0$ and $3(b+a)x^2$ if $x > 0$ and therefore we have to have $a = b$ for $x < 0$ and $a = -b$ for $x > 0$. So for arbitrary x in $(-1, 1)$ $a = b = 0$.

11. Knowing that $y(x) = x^4$ $(x > 0)$ is a solution to

$$x^2\, y'' - 12\, x\, y' + 36\, y = 0,\ x > 0,$$

use the substitution $y = x^4\, u(x)$ to find the solution to the initial value problem with $y(1) = 0$ and $y'(1) = 5$.

Solution:
$y = x^4\, u(x)$ so $y' = x^4\, u(x)' + 4x^3 u$ and $y'' = x^4 u'' + 8x^3 u' + 12x^2 u$ and therefore

$$\begin{aligned} 0 &= x^2\, y'' - 12\, x\, y' + 36\, y \\ &= x^6 u'' + 8x^5 u' + 12x^4 u - 48x^4 u - 12x^5 u' + 36x^4 u = x^4(x^2 u'' - 4xu') \\ &= x^5(xu'' - 4u') \end{aligned}$$

Therefore we must have $xu'' = 4u' = 0$ and setting $v = u'$,

$$\frac{v'}{v} = \frac{4}{x}$$

whence $v = Ax^4$ and by integrating $u = c_1 x^5 + c_2$ which yields the general solution

$$y = c_1 x^9 + c_2 x^4$$

Imposing the initial. condition we find $c_1 = 1$ and $c_2 = -1$ and

$$y = x^9 - x^4$$

12. Calculate the steady state solution for the forced damped oscillator described by the following equation for $x(t)$:

$$x'' + 4\, x' + 3\, x = 65\ \cos(2\, t)$$

Solution: The characteristic polynomial is $r^2 + 4r + 3 = (r + 3)(r + 1)$ so the steady state solution, is obtained by plugging in $x_s = A\cos(2t) + B\sin(2t)$ in the equation. We have $x_s' = -2A\sin(2t) + 2B\cos(2t)$ and $x_s'' = -4\cos(2t) - 4\sin(2t)$ and therefore

$$x_s'' + 4\, x_s' + 3\, x_s = (8B - A)cos(2t) + (8A + B)\sin(2t)$$

For $(8B - A)cos(2t) - (8A + B)\sin(2t)$ to equal $65\ \cos(2\, t)$ we must have $8B - A = 1$ and $(8A + B) = 0$ which yields $A = -1$ and $B = 8$ and so

$$x_s = -\cos(2\, t) + 8\sin(2t)$$

13. Find two values r and s $(r < s)$ for which $y_1 = x^r$ and $y_2 = x^s$ are fundamental solutions of the differential equation

$$x^2\,y'' - 18\,x\,y' + 60\,y = 0,\ x > 0.$$

Solution: Take $y = x^r$ and differentiate it twice: $y' = rx^{r-1} \qquad y'' = r(r-1)x^{r-2}$ and plug into the equation

$$0 = x^r\,(r(r-1) - 18r + 60) = x^r\left(r^2 - 19r + 60\right)$$

and so it's a solution if and only if $r^2 - 19r + 60 = (r-4)(r-15) = 0$ which means $r = 4$ and $s = 15$.