# Math 285 Midterm 2 Practice Exam

1. The existence and uniqueness theorem guarantees that there exists a unique solution of the initial value problem $(t-1)(t+3)y''' + ty' + te^{-t^2}y = \ln|t-4|$ with $y(2) = 2$, $y'(2) = 1$, and $y''(2) = 0$ on an open interval

   **A.** $(1, 4)$

   B. $(-\infty, -3)$

   C. $(-3, 1)$

   D. $(4, \infty)$

   E. None of these

> **Solution:** After dividing by $(t-1)(t+3)$, the coefficients are continuous if $t \neq 1, -3, 4$. Since the initial conditions are given at $t = 2$, the interval is $(1, 4)$.

2. If the general solution of $y'' + py' + qy = 0$ is $y(t) = e^{3t}(C_1 \cos 2t + C_2 \sin 2t)$, what are the value of $p$ and $q$?

   A. $p = -6$ and $q = 5$

   **B.** $p = -6$ **and** $q = 13$

   C. $p = 6$ and $q = 13$

   D. $p = 6$ and $q = 5$

   E. None of these

> **Solution:** Since the roots for the characteristic equation are $\lambda = -2 \pm i$, the general solution is
>
> $$y(t) = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

3. The initial value problem

$$u'' + 4u = 3\cos t$$

with $u(0) = u'(0) = 0$ has the solution $u(t) = \cos t - \cos 2t$. This can be written as a product of trigonometric functions

   A. $2\cos\left(\frac{3}{2}t\right)\sin\left(\frac{1}{2}t\right)$

   B. $\sin\left(\frac{3}{2}t\right)\sin\left(\frac{1}{2}t\right)$

   **C.** $2\sin\left(\frac{3}{2}t\right)\sin\left(\frac{1}{2}t\right)$

D. $\sin\left(\frac{3}{2}t\right)\cos\left(\frac{1}{2}t\right)$

E. None of these

**Solution:** This follows from

$$\cos(\omega t) - \cos(\omega_0 t) = 2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega_0-\omega}{2}t\right).$$

4. Identify the correct form of a particular solution for the following differential equation.

$$y'' + 2y' = te^{2t} + 10t^3.$$

A. $Y(t) = At^2e^{2t} + Bte^{2t} + C + Dt + Et^2 + Ft^3$

B. $Y(t) = Ate^{2t} + Be^{2t} + C + Dt + Et^2 + Ft^3$

C. $Y(t) = At^2e^{2t} + Bte^{2t} + Ct + Dt^2 + Et^3 + Ft^4$

**D.** $Y(t) = Ate^{2t} + Be^{2t} + Ct + Dt^2 + Et^3 + Ft^4$

E. None of these

**Solution:** Since the homogeneous equation $y'' + 2y' = 0$ has solutions $1, e^{-2t}$, a particular solution is of the form

$$Y(t) = Ate^{2t} + Be^{2t} + Ct + Dt^2 + Et^3 + Ft^4.$$

5. For which value of $k$ is the following oscillator in resonance?

$$u'' + ku = 3\cos(2t).$$

A. $k = 1$

B. $k = 2$

C. $k = 9$

D. None of these

**E.** $k = 4$

**Solution:** The oscillator is in resonance if

$$\omega_0 = \sqrt{\frac{k}{m}} = \omega.$$

6. The differential equation

$$u'' + 5u' + 9u = 0$$

corresponds to an oscillator that is

A. undamped

**B. underdamped**

C. overdamped

D. critically damped

E. None of these

**Solution:** The characteristic equation $\lambda^2 + 5\lambda + 9 = 0$ has complex roots.

7. The function $u(t) = -3\sqrt{3}\cos t + 3\sin t$ can be written as $u(t) = R\cos(\omega_0 t - \delta)$ where

A. $R = 6$, $\omega_0 = 1$, $\delta = 11\pi/6$

B. $R = 3$, $\omega_0 = 1$, $\delta = \pi/6$

C. $R = 3$, $\omega_0 = 1$, $\delta = 7\pi/6$

**D.** $R = 6$**,** $\omega_0 = 1$**,** $\delta = 5\pi/6$

E. None of these

**Solution:** It is easy to check that $R = \sqrt{(-3\sqrt{3})^2 + 3^2} = 6$ and $\omega_0 = 2$. If $\cos(\delta) = -\frac{\sqrt{3}}{2}$ and $\sin(\delta) = 1/2$, then $\delta = 5\pi/6$.

8. Which of these is NOT a set of linearly independent solutions?

**A.** $e^x, 2e^x - e^{2x}, e^x + 3e^{2x}$

B. $x, x^2 + x, 2x^3$

C. $\cos x, \sin x, \cos 2x$

D. $2, x, x\ln x$

E. None of these

**Solution:** The set $\{e^x, 2e^x - e^{2x}, e^x + 3e^{2x}\}$ is linearly dependent because

$$k_1 e^x + k_2(2e^x - e^{2x}) + k_3(e^x + 3e^{2x}) = 0$$

holds for all $x$ if $k_1 = -7$, $k_2 = 3$, and $k_3 = 1$.

9. The third order differential equation

$$y''' + py'' + qy' + ry = 0$$

has the characteristic equation $(\lambda + 1)(\lambda^2 + 4\lambda + 5) = 0$. What is the general solution to the differential equation?

   A. $C_1 e^{-t} + C_2 e^{t} + C_3 e^{-5t}$
   B. $C_1 e^{t} + C_2 e^{-2t}\cos t + C_3 e^{-2t}\sin t$
   **C.** $C_1 e^{-t} + C_2 e^{-2t}\cos t + C_3 e^{-2t}\sin t$
   D. $C_1 e^{t} + C_2 e^{t} + C_3 e^{-5t}$
   E. None of these

**Solution:** Since the roots are $\lambda = -1, -2 \pm i$. Thus, the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t}\cos t + C_3 e^{-2t}\sin t.$$

10. Find the solution to the following initial value problem

$$y'' - 6y' + 9y = 2te^{3t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

**Solution:** The general solution of $y'' - 6y' + 9y = 0$ is $C_1 e^{3t} + C_2 te^{3t}$. By the method of undetermined coefficients, a particular solution has a form

$$Y(t) = t^2(At + B)e^{3t} = (At^3 + Bt^2)e^{3t}.$$

We then have

$$\begin{aligned} Y'' - 6Y' + 9Y &= ((6At + 2B)e^{3t} + 6(3At^2 + 2Bt)e^{3t} + 9(At^3 + Bt^2)e^{3t}) \\ &\quad - 6((3At^2 + 2Bt)e^{3t} + 3(At^3 + Bt^2)e^{3t}) + 9(At^3 + Bt^2)e^{3t} \\ &= (6At + 2B)e^{3t} \\ &= 2te^{3t}, \end{aligned}$$

which means $A = \frac{1}{3}$ and $B = 0$. Thus, the solution is

$$y(t) = C_1 e^{3t} + C_2 t e^{3t} + \frac{1}{3} t^3 e^{3t}.$$

By the initial conditions,

$$y(t) = e^{3t} - 3te^{3t} + \frac{1}{3} t^3 e^{3t}.$$

11. Consider $t^2 y'' - 5ty' + 9y = 0$ for $t > 0$.

    1. Find $r$ such that $y_1(t) = t^r$ is a solution to the equation.
    2. Find another solution $y_2$ to the equation such that $W[y_1, y_2](t) \neq 0$. (Hint: use the method of reduction of order.)

**Solution:**

1. Since

$$\begin{aligned} t^2 y_1'' - 5ty_1' + 9y_1 &= r(r-1)t^r - 5rt^r + 9t^r \\ &= t^r(r^2 - 6r + 9) \\ &= 0, \end{aligned}$$

   we obtain $r = 3$.

2. Let $y_2 = vy_1$, then

$$\begin{aligned} t^2 y_2'' - 5ty_2' + 9y_2 &= t^2(v''y_1 + 2v'y_1' + vy_1'') - 5t(v'y_1 + vy_1') + 9vy_1 \\ &= t^5 v'' + t^4 v' \\ &= t^4(tv'' + v') \\ &= 0. \end{aligned}$$

   Thus, $tv'' + v' = 0$. By solving the equation, we get $v(t) = C_1 \ln t + C_2$. Let $y_2 = t^3 \ln t$, then $y_2$ is a solution and

$$\begin{aligned} W[y_1, y_2] &= t^3(3t^2 \ln t + t^2) - 3t^2(t^3 \ln t) \\ &= t^5. \end{aligned}$$