Solutions to Midterm 1

1. If $v(x) = \frac{y(x)}{x^2}$, then

$$xy' = 2y + 3y^2 + 6x^4$$

can be transformed into

A. $v' = 3x(v^2 + 2)$

B. $v' = 3v^2 + 6$

C. $xv' = 2v + 3x^2(v^2 + 2)$

D. None of these

E. $xv' = 3v^2 + 6x^2$

Solution: Since $v(x) = \frac{y(x)}{x^2}$, we have $y = x^2 v$ and $y' = x^2 v' + 2xv$. So, the given equation can be written as

$$xy' = x^3 v' + 2x^2 v = 2x^2 v + 3x^4 v^2 + 6x^4 = 2y + 3y^2 + 6x^4.$$

Dividing x^2 both sides, we get

$$xv' + 2v = 2v + 3x^2 v^2 + 6x^2 = 2v + 3x^2(v^2 + 2)$$

as desired.

2. Determine the stability of the equilibrium of the following equation for $P(t)$.

$$\frac{dP}{dt} = 2P(P + 3)(P - 1).$$

A. $P(t) = -3$ **and** $P(t) = 1$ **are unstable and** $P(t) = 0$ **asymptotically stable.**

B. $P(t) = -3$, $P(t) = 1$, $P(t) = 0$ are unstable.

C. $P(t) = -3$ and $P(t) = 1$ are asymptotically stable and $P(t) = 0$ is unstable.

D. None of these.

E. $P(t) = -3$ and $P(t) = 0$ are unstable and $P(t) = 1$ asymptotically stable.

Solution: Let $f(P) = 2P(P+3)(P-1)$, then $f(P) > 0$ if $P \in (-3, 0) \cup (1, \infty)$ and $f(P) < 0$ if $P \in (-\infty, -3) \cup (0, 1)$. Thus, $P(t) = -3$ and $P(t) = 1$ are unstable and $P(t) = 0$ asymptotically stable.

3. The Wronskian of $y_1(t) = e^{-3t^2-3t+6}$ and $y_2(t) = e^{-3t^2-3t-3}$ is

   A. 0

   B. None of these.

   C. $e^{-6t^2-6t+12}(-12t-6)$

   D. $9e^{-3t^2-3t+6}$

   E. $e^9(-12t-6)$

Solution: Note that $y_1(t) = e^9 y_2(t)$. If y_1 is a scalar multiple of y_2, then $W[y_1, y_2](t) = 0$ for all t. Indeed,

$$W[y_1, y_2](t) = y_1 y_2' - y_1' y_2 = e^9 y_2 y_2' - e^9 y_2 y_2' = 0.$$

4. If $y(t) = 5e^{-4t} - 4e^{5t}$ is a solution to the following equation

$$y'' + py' + qy = 0,$$

what are p and q?

Solution: Since $y(t)$ is a solution, we have

$$\begin{aligned} y'' + py' + qy &= (5e^{-4t} - 4e^{5t})'' + p(5e^{-4t} - 4e^{5t})' + q(5e^{-4t} - 4e^{5t}) \\ &= 5e^{-4t}(16 - 4p + q) - 4e^{5t}(25 + 5p + q) \\ &= 0 \end{aligned}$$

for all t. This implies that $16 - 4p + q = 25 + 5p + q = 0$. Therefore, $p = -1$ and $q = -20$.

5. Consider

$$y' = \frac{(y-5)(y+5)}{y(y-8)}.$$

Solution: Let $f(y) = \frac{(y-5)(y+5)}{y(y-8)}$, then $f(y) > 0$ if $y \in (-\infty, -5) \cup (0, 5) \cup (8, \infty)$ and $f(y) < 0$ if $y \in (-5, 0) \cup (5, 8)$. If $y(0) = -1$, then $y(t)$ is decreading so that $\lim_{t\to\infty} y(t) = -5$. If $y(0) = 2$, then $y(t)$ is increasing so that $\lim_{t\to\infty} y(t) = 5$.