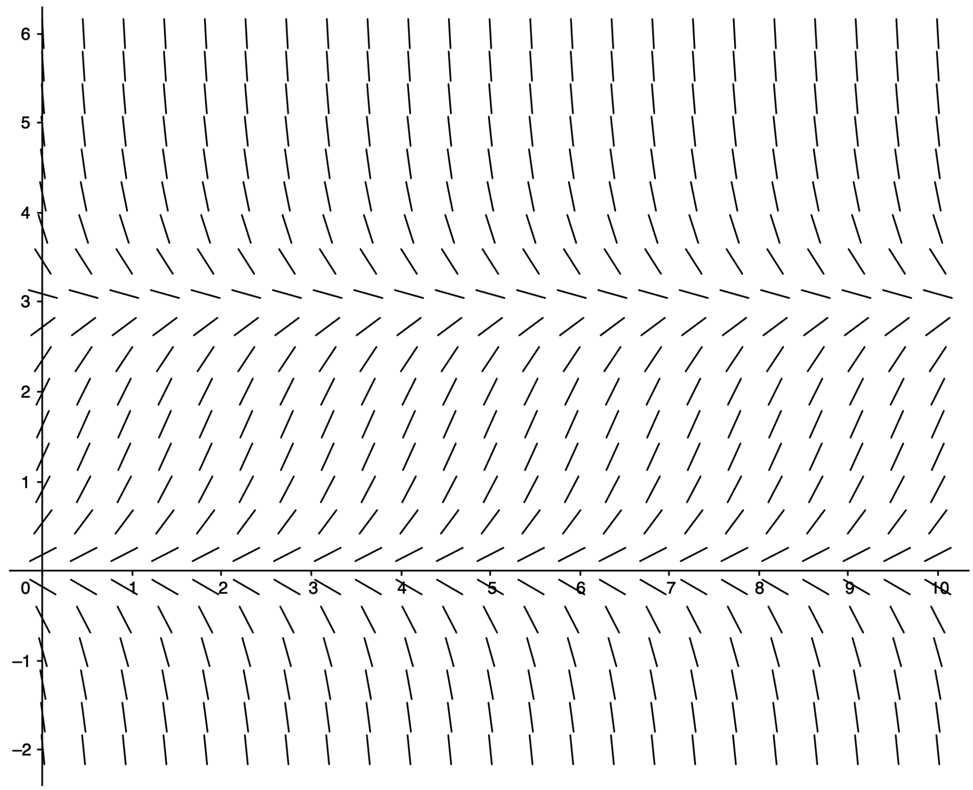
Math 285 Midterm 1 Practice Exam

1. Identify which differential equation below corresponds to the following direction field.



A. $y' = y(3-y)$
B. $y' = y(y-3)$
C. $y' = y(y-2)$
D. $y' = y-3$
E. None of these

Solution: The slopes are zero if $y = 0, 3$. If $y > 3$ or $y < 0$, then the slope is negative. If $0 < y < 3$, then the slope is positive. Thus, $y' = -y(y-3)$.

2. Find the solution of

$$\frac{dy}{dt} = 5 - 2y.$$

A. $y(t) = Ce^{-2t} + \frac{5}{2}$ **where** C **is an arbitrary constant.**

B. $y(t) = Ce^{-t} + \frac{5}{2}$ where C is an arbitrary constant.

C. $y(t) = e^{-2t} + C$ where C is an arbitrary constant.

D. $y(t) = e^{-t} + C$ where C is an arbitrary constant.

E. None of these

Solution: Since it is separable,

$$\frac{1}{2y-5}dy = -dt$$

and so $\frac{1}{2}\ln|2y-5| = -t + C$. Thus, $y(t) = Ce^{-2t} + \frac{5}{2}$.

3. How would you classify the following equation for $y(t)$?

$$(1+t^2)(y''')^2 + e^t \ln y = \frac{y}{t}.$$

A. A third order nonlinear differential equation.

B. A third order linear differential equation.

C. A sixth order nonlinear differential equation.

D. A sixth order linear differential equation.

E. None of these

Solution: The highest derivative is y''' and there are nonlinear terms $(y''')^2$ and $\ln y$.

4. What is the correct integrating factor to solve the following ODE for $y(t)$?

$$t^4 y' + 3t^3 y + 1 = 0.$$

A. $\mu(t) = t^3$

B. $\mu(t) = \frac{1}{t^3}$

C. $\mu(t) = 3t$

D. $\mu(t) = \frac{3}{t}$

E. None of these

Solution: Dividing the equation by t^4, we get $p(t) = 3/t$ and

$$\mu(t) = \exp(3 \int \frac{1}{t}\, dt) = t^3.$$

5. Solve the following initial value problem.

$$y' + 3y = e^{-t}, \qquad y(1) = 0.$$

A. $y(t) = \frac{1}{2}(e^{-t} - e^{2-3t})$

B. $y(t) = -e^{-4t} + e^{-3t-1}$

C. $y(t) = \frac{1}{2}(e^{2t} - e^2)$

D. $y(t) = e^{-t} - e^{-1}$

E. None of these

Solution: Since the integrating factor is $\mu(t) = e^{3t}$, we have

$$y(t) = e^{-3t} \int e^{2t} dt = e^{-3t}(\frac{1}{2}e^{2t} + C).$$

Since $y(1) = 0$, the solution is

$$y(t) = \frac{1}{2}(e^{-t} - e^{2-3t}).$$

6. The following nonlinear equation for $y(x)$ can be transformed with a substitution into which separable equation for $v(x)$?

$$y' = \frac{-x^2 + 2xy + y^2}{x^2} + \sec\left(\frac{y}{x}\right)$$

A. $xv' = -1 + v + v^2 + \sec v$

B. $v' = -1 + 2v + v^2 + \sec v$

C. $xv' = -x^2 + v + v^2 + \sec v$

D. $v' = 1 - 2v + v^2$

E. None of these

Solution: Let $v = y/x$, then $v + xv' = y'$ and the RHS is $-1 + 2v + v^2 + \sec v$.

7. Determine the values of r for which the given differential equation has solutions of the form $y(t) = e^{rt}$.

$$y'' + 4y' - 5y = 0.$$

A. $r = 1, -5$

B. $r = -1, 5$

C. $r = 1, 5$

D. $r = 3, -5$

E. None of these

Solution: Plugging $y(t) = e^{rt}$ into the equation, we get $(r^2 + 4r - 5)e^{rt} = 0$. Since $e^{rt} \neq 0$, we have $r = 1, -5$.

8. The existence and uniqueness theorem for linear differential equations ensures that the solution of

$$(t-2)(t+3)y' + \sqrt{t}y = 1 + t^2, \qquad y(1) = -1$$

exists for all t in the region defined by:

A. $(0, 2)$

B. $(-3, 2)$

C. $(2, \infty)$

D. $(-\infty, -3)$

E. None of these

Solution: Dividing the equation by $(t-2)(t+3)$, we have

$$y' + \frac{\sqrt{t}}{(t-2)(t+3)}y = \frac{1+t^2}{(t-2)(t+3)}.$$

The first coefficient is continuous on $(0, 2) \cup (2, \infty)$ and the second on $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$. Since the initial condition $1 \in (0, 2)$, the interval is $(0, 2)$.

9. Consider the following nonlinear ODE for $y(x)$. Which initial condition would guarantee a unique solution to the initial value problem?

$$yy' + x^2y' = \frac{e^{-x}}{x+1}$$

A. $y(2) = 4$
B. $y(2) = -4$
C. $y(-1) = 1$
D. $y(-1) = -1$
E. None of these

Solution: Since $yy' + x^2y' = y'(y + x^2)$, we have

$$y' = \frac{e^{-x}}{(y+x^2)(x+1)} = f(x, y).$$

Then, f and $\frac{\partial f}{\partial y}$ are continuous if $y \neq -x^2$ and $x \neq -1$.

10. For which values of r and K, is the constant function $y(t) = -2$ an asymptotic stable equilibrium solution to the following equation?

$$y' = r(y + 2)(y - K).$$

A. $r = 1$ **and** $K = 0$
B. $r = 2$ and $K = -3$
C. $r = -1$ and $K = -2$
D. $r = -2$ and $K = 1$
E. None of these

Solution: In order for $y(t) = -2$ to be asymptotic stable, $f(y) = r(y + 2)(y - K)$ is negative if y is close to -2 from right and positive if y is close to -2 from left. Thus, this holds if $r > 0$ and $K > -2$ or if $r < 0$ and $K < -2$.

11. Consider the following differential equation.

$$y' = \frac{x^4 + 3y}{x}, \qquad x > 0.$$

1. Determine whether the equation is linear. Give a reason why.
2. Solve the equation with the initial condition $y(2) = 4$.

Solution:

1. It is linear because it can be written as $y' - \frac{3}{x}y = x^3$.
2. The integrating factor is $\mu(x) = x^{-3}$. Thus,

$$y(x) = \frac{1}{\mu(t)} \int \mu(x)x^3\, dx = x^3(x + C) = x^4 + Cx^3.$$

Since $y(2) = 4$, $C = -\frac{3}{2}$ and so

$$y(x) = x^4 - \frac{3}{2}x^3.$$

12. Find the solution to the following initial value problem.

$$(x^4 + 1)y' = 2x^3y^2, \qquad y(0) = \frac{3}{2}$$

and specify the interval where the solution is defined.

Solution: The equation is separable and

$$\frac{1}{y^2}dy = \frac{2x^3}{x^4 + 1}dx.$$

Taking integration of the both sides, we have

$$-\frac{1}{y} = \frac{1}{2}\ln(x^4 + 1) + C.$$

Using the initial condition, $C = -2/3$. Thus,

$$y(x) = \frac{6}{4 - 3\ln(x^4 + 1)}.$$

Since the denominator is zero if

$$x = \pm(e^{4/3} - 1)^{\frac{1}{4}},$$

the solution is defined on the interval

$$(-(e^{4/3} - 1)^{\frac{1}{4}}, (e^{4/3} - 1)^{\frac{1}{4}}).$$