Practice for Final exam

1. Consider the following initial value problem for $y(x)$:

$$y' = \frac{\sqrt{y-a}}{(x-b)^2}, \qquad y(x_0) = y_0.$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

A. $x_0 = \beta \neq b$ and $y_0 = \alpha > a$

B. $x_0 = b$ and $y_0 = \alpha > a$

C. $x_0 = \beta \neq b$ and $y_0 = a$

D. $x_0 = b$ and $y_0 = a$

E. $x_0 = \beta \neq b$ and $y_0 = a$

F. None of these

2. Consider the following initial value problem for $y(x)$:

$$y' = \frac{x}{(y-a)(x-b)^2}, \qquad y(x_0) = y_0$$

For which values of x_0 and y_0 are we guaranteed one and only one solution?

A. $x_0 = \beta \neq b$ and $y_0 = \alpha \neq a$

B. $x_0 = b$ and $y_0 = \alpha \neq a$

C. $x_0 = \beta \neq b$ and $y_0 = a$

D. $x_0 = b$ and $y_0 = a$

E. $x_0 = \beta \neq b$ and $y_0 = a$

F. None of these

3. Select all sets of solutions that are linearly independent on the whole real axis

   A. e^x, e^{2x}, $ae^x + be^{2x}$,

   B. $\sin(2x)$, $\cos(x)\sin(x)$, x,

   C. x, x^2, $ax^2 + bx$,

   D. $x + 1$, $x - 1$, 1,

   E. e^x, e^{2x}, e^{3x},

   F. $\sin(2x)$, $\cos(2x)$, 1,

   G. $x + 1$, $x - 1$, x^2,

   H. 1, $ae^x, +b\,e^{2x}$

   I. $e^x + 1$, $e^{-x} - 1, x$

4. Assume that a linear homogeneous ODE for $y(x)$ has one of the following characteristic equation

   (i) $r^3 - 4r^2 + 4r$

   (ii) $r^3 - 2r^2$

   (iii) $r^3 + 6r^2 + 9r$

   (iv) $r^3 + 3r^2$

   (v) $r^4 + r^3 - 6r^2$

   In each case pare item (i)-(v) above with a solution below

   A. $c_1 + c_2 e^{2x} + c_3 x e^{2x}$

   B. $c_1 + c_2 x + c_3 e^{2x}$

   C. $c_1 + c_2 e^{-3x} + c_3 x e^{-3x}$

   D. $c_1 + c_2 x + c_3 e^{-3x}$

   E. $c_1 + c_2 x + c_3 e^{2x} + c_4 e^{-3x}$

   F. $c_1 + c_2 x + c_3 x^2$

   G. $c_1 e^{2x} + c_2 e^{-3x}$

   H. $c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-3x}$

   I. $c_1 e^{2x} + c_2 e^{-3x} + c_3 x e^{-3x}$

5. Consider the following initial value problem for $y(x)$:

$$(x^2 - (a+b)\,x\; + ab)\,y' + \frac{\gamma\, x}{x^2 - 2c\,x + c^2}\,y = \gamma_2\,\frac{x^2 + 2c\,x + c^2}{x^2}$$

with $y(x_0) = \gamma_2$. Assuming that $a < 0 < b < c$, determine the interval in which we are guaranteed one and only one solution:

$$x \in \Big(\;,\;\Big).$$

6. Determine the integrating factor for the following ODE for $y(x)$:

$$x^2\, y' + A\, x\, y = Bx^C, \qquad x > 0$$

7. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = \gamma(P^2 - (a+b)P + ab)(P^2 - 2c\,P + c^2)$$

with $c < 0 < a < b$. Identify the correct equilibrium solutions and their stability

A. if $\gamma > 0$ $P = c$, semistable; $P = a$, stable; $P = b$, unstable

B. if $\gamma < 0$ $P = c$, semistable; $P = a$, unstable; $P = b$, stable

C. $P = c$, stable; $P = a$, unstable; $P = b$, stable

D. $P = c$, stable; $P = a$, stable; $P = b$, unstable

E. $P = 0$, semistable; $P = a$, stable; $P = b$, unstable

F. $P = 0$, semistable; $P = a$, unstable; $P = b$, stable

G. $P = c$, stable; $P = 0$, unstable; $P = b$, stable

H. $P = c$, stable; $P = 0$, stable; $P = b$, unstable

I. $P = c$, stable; $P = a$, unstable; $P = 0$, stable

J. $P = c$, unstable; $P = a$, stable; $P = 0$, unstable

K. None of these

8. Consider the population model for $P(t)$ described by the ODE:

$$\frac{dP}{dt} = (P^4 + (b-a)P^3 - ab\,P^2)$$

with $b > a > 0$. Determine the value of the stable equilibrium solution $P =$. Determine the value of the unstable equilibrium solution $P =$.

9. Consider the following oscillator equation for $x(t)$ and the given initial conditions:

$$x'' + \omega_0^2\, x = F_0\, f(\omega t),\ x(0) = 0;\ x'(0) = 0$$

with $\omega \neq \omega_0$ and where either $f = \cos$ or $f = \sin$. What is the long term behavior of the solution?

A. The solution will oscillate forever

B. The solution is 0 at all times

C. The solution will oscillates with amplitude growing to infinity

D. The solution will decay to 0

E. The solution will oscillates with amplitude decaying to 0

F. There is no solution

G. The solution will reach a finite asymptote

H. The solution will be lost in a forest

I. None of these

10. Consider the following ODE's for $y(x)$

   (i) $y'' - y' - 6\,y = -4\,e^{x} + 3\,e^{-2x}$

   (ii) $y'' + 3\,y' - 4\,y = 2\,e^{-2x} - e^{-4x}$

   (iii) $y'' + y' - 6\,y = e^{2x} + 4\,e^{-2x}$

   (iv) $y'' - 2\,y' - 8\,y = 2\,e^{4x} - 4\,e^{2x}$,

   (v) $y'' + 2\,y' - 3\,y = 2\,e^{x} + e^{-4x}$

   If you were to use the method of variation of parameters, what would be the correct particular solution to use?

   A. $u_1(x)\,e^{3x} + u_2(x)\,e^{-2x}$ **for item 1**

   B. $u_1(x)\,e^{x} + u_2(x)\,e^{-4x}$ **for item 2**

   C. $u_1(x)\,e^{2x} + u_2(x)\,e^{-3x}$ **for item 3**

   D. $u_1(x)\,e^{4x} + u_2(x)\,e^{-2x}$ **for item 4**

   E. $u_1(x)\,e^{x} + u2(x)\,e^{-3x}$ **for item 5**

   F. $A\,e^{x} + B\,e^{-2x}$

   G. $A\,e^{-2x} + B\,e^{-4x}$

   H. $A\,e^{2x} + B\,e^{-2x}$

   I. $A\,e^{4x} + B\,e^{2x}$

   J. $A\,e^{x} + B\,e^{-4x}$

   K. $A\,e^{x} + B\,x\,e^{-2x}$

   L. $A\,e^{-2x} + B\,x\,e^{-4x}$

   M. $A\,x\,e^{2x} + B\,e^{-2x}$

   N. $A\,x\,e^{4x} + B\,e^{2x}$

   O. $A\,x\,e^{x} + B\,e^{-4x}$

11. Consider the following eigenvalue problem for $y(x)$

$$y'' + \lambda\, y = 0,$$

with

$$y(0) - y'(0) = 0, \qquad y(L) = 0, \tag{A}$$
$$y(0) = 0, \qquad y(L) + y'(L) = 0, \tag{B}$$
$$y'(0) = 0, \qquad y(L) + y'(L) = 0. \tag{C}$$

Which are the correct eigenvalues and corresponding eigenfunctions?

A. **First choice of boundary conditions** $\lambda_n = \alpha_n^2, \qquad y_n(x) = \alpha_n \cos(\alpha_n x) + \sin(\alpha_n x)$ where $\tan(L\,\alpha_n) = -\alpha_n$

B. **Second choice of boundary conditions**$\lambda_n = \alpha_n^2, \qquad y_n(x) = \sin(\alpha_n x)$, where $\tan(L\,\alpha_n) = -\alpha_n$

C. **Third choice of boundary conditions** $\lambda_n = \alpha_n^2, \qquad y_n(x) = \cos(\alpha_n x)$, where , $\tan(L\,\alpha_n) = 1/\alpha_n$

D. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

E. $\lambda_n = \frac{n^2\pi^2}{L^2}; \qquad y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

F. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}; \qquad y_n(x) = \cos\left(\frac{(2n-1)\pi x}{L}\right)$

G. $\lambda_n = \frac{(2n-1)^2\pi^2}{L}; \qquad y_n(x) = \sin\left(\frac{(2n-1)\pi x}{L}\right)$

H. None of these

12. Calculate the coefficients of the Cosine Fourier Series expansion of $f(x) = a\,x$ for $0 < x < L$.

13. Using separation of variables solve the following diffusion equation problem for $u(x,t)$ where $0 < x < L$ and $t > 0$

$$\begin{cases} u_t = \kappa\, u_{xx} & \text{for } 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, & \text{for } t \geq 0, \\ u(x,0) = a\, x. & \end{cases}$$

Assume that the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n\, T_n(t)\, X_n(x).$$

Calculate c_n, $T_n(t)$, and $X_n(x)$.

14. Consider the Laplace equation problem in the rectangle $0 < x < a$ and $0 < y < b$:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, \\ &u(0,y) = bc_1, \qquad u(a,y) = bc_2, \\ &u(x,0) = bc_3, \qquad u(x,b) = bc_4. \end{aligned}$$

Where bc_i means that the i-th condition is a function $f(x)$ and everything else is 0. If you were to solve this problem by separation of variables by writing $u(x,y) = X(x)Y(y)$, what would be the solutions for X_n and Y_n?

A. **Correct in case** bc_1 $X_n = -\tanh\left(\frac{an\pi}{b}\right)\cosh\left(\frac{n\pi x}{b}\right) + \sinh\left(\frac{n\pi x}{b}\right)$; $Y_n = \sin\left(\frac{n\pi y}{b}\right)$;

B. **Correct in case** bc_2 $X_n = \sinh\left(\frac{n\pi x}{b}\right)$; $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

C. **Correct in case** bc_3 $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $Y_n = -\tanh\left(\frac{bn\pi}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) + \sinh\left(\frac{n\pi y}{a}\right)$

D. **Correct in case** bc_4 $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $Y_n = \sinh\left(\frac{n\pi y}{a}\right)$

E. None of these

F. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$; $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

G. $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $Y_n = \cos\left(\frac{n\pi y}{b}\right)$

H. $X_n = \tan\left(\frac{bn\pi}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$; $Y_n = \cos\left(\frac{n\pi y}{b}\right)$

I. $X_n = \sin\left(\frac{n\pi x}{a}\right)$; $Y_n = \sin\left(\frac{n\pi y}{b}\right)$

J. There is no solution