Math 3215: Intro to Probability and Statistics

Exam 2, Summer 2023

Date: July 6, 2023

NAME:______________________________ ID:______________________________

READ THE FOLLOWING INFORMATION.

- This is a 75-minute.
- This exam contains 10 pages (including this cover page) and 7 questions. Total of points is 100.
- Books, notes, and other aids are not allowed.
- Show all steps to earn full credit.
- Do not unstaple pages. Loose pages will be ignored.

Name	**PMF**	**Mean**	**Variance**
Ber(p)	$\mathbb{P}(X=1)=p, \mathbb{P}(X=0)=1-p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$ for $x=0,1,\ldots,n$	np	$np(1-p)$
Geom(p)	$p(1-p)^{x-1}$ for $x=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\binom{x-1}{r-1}p^r q^{x-r}$ for $x=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}$ for $x=0,1,\ldots$	λ	λ
Uniform(a,b)	$\frac{1}{b-a}$ for $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ for $x\in(-\infty,\infty)$	μ	σ^2
Exp(λ)	$\lambda e^{-\lambda x}$ for $x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(a,λ)	$\frac{\lambda^a x^{a-1}e^{-\lambda x}}{\Gamma(a)}$ for $x>0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$

1. Let X and Y be two discrete random variables with joint pmf

$$f_{X,Y}(1,1) = f_{X,Y}(2,1) = \frac{1}{8}, \quad f_{X,Y}(1,2) = \frac{1}{4}, \quad f_{X,Y}(2,2) = \frac{1}{2}.$$

   (a) (5 points) Find $\mathbb{E}[XY]$.

   (b) (5 points) Find the conditional expectation of X given $Y = 1$.

   (c) (5 points) Find the conditional expectation $\mathbb{E}[X|Y]$.

2. Let X and Y be continuous random variables with joint probabilitydensity function

$$f(x,y) = \frac{x}{5} + cy$$

for $0 < x < 1$ and $1 < y < 5$, and otherwise 0.

(a) (5 points) Find the constant c.

(b) (5 points) Find the marginal pdfs of X and Y.

(c) (5 points) Are they independent?

3. Let X be a random variable with cdf given by

$$F_X(x) = 1 - e^{-e^x}$$

for $-\infty < x < \infty$. Let $Y = e^X$.

(a) (7 points) Find the pdf of X.

(b) (8 points) Find the cdf and pdf of Y.

4. Let X be a uniform random variable on $(-1,1)$ and $Y = |X|$.
   (a) (5 points) Find the pdf of Y.

   (b) (5 points) Compute $\text{Cov}(X,Y)$.

   (c) (5 points) Compute $\mathbb{P}(X \leq \frac{1}{2})$, $\mathbb{P}(Y \leq \frac{1}{2})$, and $\mathbb{P}(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$.

5. Let (X,Y) have a bivariate normal distribution with common mean 24, common standard deviation $2\sqrt{3}$, and correlation coefficient 0.5. That is, $\mu_X = \mu_Y = 24$, $\sigma_X = \sigma_Y = 2\sqrt{3}$, and $\rho = 0.5$.

   (a) (7 points) Find the expectation of the variance of $X+Y$.

   (b) (8 points) Using the tables, find the conditional probability $\mathbb{P}(X \geq 24.75 | Y = 12)$.

6. Let X be a normal random variable with mean 5 and variance 4, that is, $X \sim N(5,4)$

   (a) (5 points) Find $\mathbb{E}[(3X-2)^2]$.

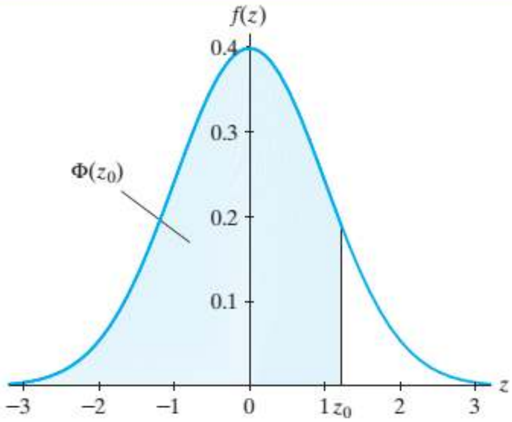
   (b) (5 points) Using the tables, find $\mathbb{P}(X \leq 3.5)$.

7. Let X, Y be independent exponential random variables with parameters $\lambda_X = 1$ and $\lambda_Y = 2$, that is, their marginal pdfs are $f_X(t) = e^{-t}$ and $f_Y(t) = 2e^{-2t}$ for $t \geq 0$.

   (a) (7 points) Let $Z = \max\{X, Y\}$. Find $\mathbb{P}(Z \leq 6)$.

   (b) (8 points) Let $W = \min\{X, Y\}$. Find the cdf and the pdf of W.

Table Va The Standard Normal Distribution Function



$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$
$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.240	2.326	2.576	2.807	3.291

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