# Section 3.1 : Introduction to Determinants

Chapter 3 : Determinants

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. The definition and computation of a determinant
2. The determinant of triangular matrices

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Compute determinants of $n \times n$ matrices using a cofactor expansion.
2. Apply theorems to compute determinants of matrices that have particular structures.

# A Definition of the Determinant

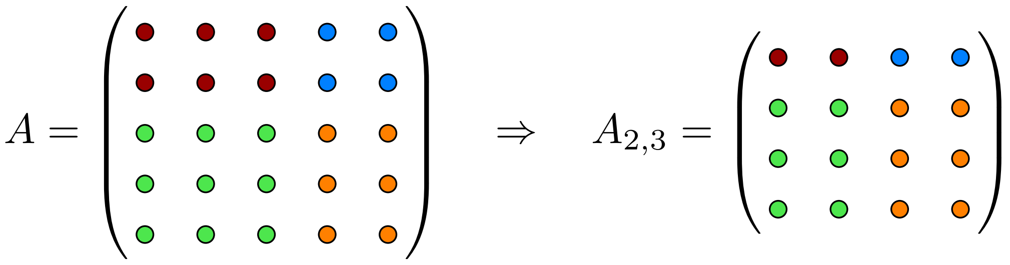
Suppose $A$ is $n \times n$ and has elements $a_{ij}$.

1. If $n = 1$, $A = [a_{11}]$, and has determinant $\det A = a_{11}$.
2. Inductive case: for $n > 1$,

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

where $A_{ij}$ is the submatrix obtained by eliminating row $i$ and column $j$ of $A$.

**Example**



# Example 1

Compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

# Example 2

Compute $\det \begin{bmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{vmatrix} 1 & -5 & 0 \\ 2 & 4 & -1 \\ 0 & 2 & 0 \end{vmatrix}$.

# Cofactors

Cofactors give us a more convenient notation for determinants.

**Definition: Cofactor**

The $(i, j)$ cofactor of an $n \times n$ matrix $A$ is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

The pattern for the negative signs is

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

**Theorem**

The determinant of a matrix $A$ can be computed down any row or column of the matrix. For instance, down the $j^{th}$ column, the determinant is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This gives us a way to calculate determinants more efficiently.

# Example 3

Compute the determinant of $\begin{bmatrix} 5 & 4 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix}$.

# Triangular Matrices

**Theorem**

If $A$ is a triangular matrix then

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

**Example 4**

Compute the determinant of the matrix. Empty elements are zero.

$$\begin{bmatrix} 2 & 1 & & & & & \\ & 2 & 1 & & & & \\ & & 2 & 1 & & & \\ & & & 2 & 1 & & \\ & & & & 2 & 1 & \\ & & & & & 2 & 1 \\ & & & & & & 2 \end{bmatrix}$$

# Computational Efficiency

Note that computation of a co-factor expansion for an $N \times N$ matrix requires roughly $N!$ multiplications.

- A $10 \times 10$ matrix requires roughly $10! = 3.6$ million multiplications
- A $20 \times 20$ matrix requires $20! \approx 2.4 \times 10^{18}$ multiplications

Co-factor expansions may not be practical, but determinants are still useful.

- We will explore other methods for computing determinants that are more efficient.
- Determinants are very useful in multivariable calculus for solving certain integration problems.

# Section 3.2 : Properties of the Determinant

Chapter 3 : Determinants

Math 1554 Linear Algebra

*"A problem isn't finished just because you've found the right answer."*
- Yōko Ogawa

We have a method for computing determinants, but without some of the strategies we explore in this section, the algorithm can be very inefficient.

# Topics and Objectives

**Topics**

We will cover these topics in this section.

- The relationships between row reductions, the invertibility of a matrix, and determinants.

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Apply properties of determinants (related to row reductions, transpose, and matrix products) to compute determinants.
2. Use determinants to determine whether a square matrix is invertible.

# Row Operations

- We saw how determinants are difficult or impossible to compute with a cofactor expansion for large $N$.
- Row operations give us a more efficient way to compute determinants.

**Theorem: Row Operations and the Determinant**

Let $A$ be a square matrix.

1. If a multiple of a row of $A$ is added to another row to produce $B$, then $\det B = \det A$.
2. If two rows are interchanged to produce $B$, then $\det B = -\det A$.
3. If one row of $A$ is multiplied by a scalar $k$ to produce $B$, then $\det B = k \det A$.

**Example 1** Compute $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix}$

# Invertibility

Important practical implication: If $A$ is reduced to echelon form, by $r$ interchanges of rows and columns, then

$$|A| = \begin{cases} (-1)^r \times \text{(product of pivots)}, & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is singular.} \end{cases}$$

**Example 2** Compute the determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & 2 \end{vmatrix}$$

# Properties of the Determinant

For any square matrices $A$ and $B$, we can show the following.

1. $\det A = \det A^T$.
2. $A$ is invertible if and only if $\det A \neq 0$.
3. $\det(AB) = \det A \cdot \det B$.

⇒ If $AB$ invertible

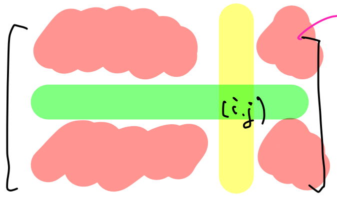
⇒ $A$, $B$ invertible

Determinant  Det : $n \times n$ matrix $\longmapsto$ a real number

- Cofactor Expansion

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad C_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$$

A =



$A_{ij}$ : $(n-1) \times (n-1)$ matrix

removing $i$th row $j$th col.

from $A$.

- If $A$ is triangular, $\det(A)$ = product of diagonal entries

$$A \xrightarrow{\text{row operation}} \underline{\text{Echelon form}}$$

replacement $(R_2 \to R_2 - 7 \cdot R_1)$ doesn't change det.

interchange $(R_2 \leftrightarrow R_3)$ det → $(-1) \cdot$ det

scaling $(R_4 \to 7 \cdot R_4)$ det → $7 \cdot$ det.

$$C_{ij} = \underline{\underline{(-1)^{i+j}}} \det(A_{ij})$$

# Additional Example (if time permits)

Use a determinant to find all values of $\lambda$ such that matrix $C$ is not invertible.

$\Updownarrow$

$\det C = 0$

$$C = \overbrace{\begin{pmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}^{'' A} - \lambda I_3 = A - \lambda \cdot I_3$$

$$= \begin{pmatrix} \overset{(1,1)}{5-\lambda} & \overset{(1,2)}{\boxed{0}} & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix}$$

$$\det(C) = (5-\lambda)\cdot(-1)^{1+1} \cdot \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} + \underline{\underline{0}} \cdot (-1)^{1+2} \det\begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$+ 0 \cdot \;---$$

$$= (5-\lambda)\cdot 1 \cdot \left( (-\lambda)\cdot(-\lambda) - 1\cdot 1 \right)$$

$$= (5-\lambda)(\lambda^2 - 1) = (5-\lambda)(\lambda+1)(\lambda-1) = 0$$

$\lambda = 5, 1, -1$ ← Eigenvalues for A

$C$ is not invertible $\Longleftrightarrow$ $C\vec{x} = \vec{0}$ has nontrivial solutions

$\vec{x} \neq \vec{0}$ & $\underline{\underline{C}}\,\vec{x} = \vec{0}$

$(A \underbrace{- \lambda I}) \vec{x} = \vec{0}$

$A\vec{x} - \lambda\vec{x} = 0$ → $A\vec{x} = \lambda\vec{x}$ ↗ Diagonalization ↘

$A^{10}\vec{x} = A^9 \cdot (A\vec{x}) = A^9 \cdot (\lambda\vec{x}) = \lambda A^8 \cdot (\underline{A\vec{x}}) = \cdots = \lambda^{10} \cdot \vec{x}$

# Additional Example (if time permits)

Determine the value of

$$\det A = \det \left( \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}^8 \right).$$

# Section 3.3 : Volume, Linear Transformations

Chapter 3 : Determinants

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. Relationships between area, volume, determinants, and linear transformations.

**Objectives**

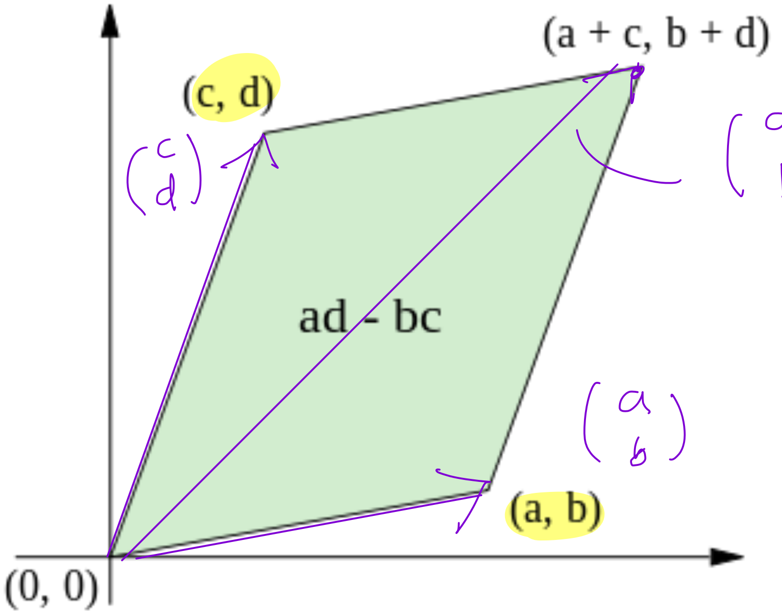
For the topics covered in this section, students are expected to be able to do the following.

1. Use determinants to compute the area of a parallelogram, or the volume of a parallelepiped, possibly under a given linear transformation.

Students are not expected to be familiar with Cramer's rule.

# Determinants, Area and Volume

In $\mathbb{R}^2$, determinants give us the area of a parallelogram.



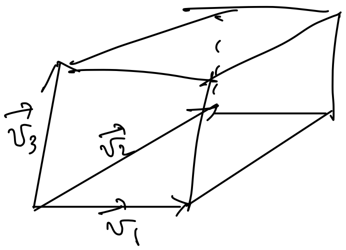
$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

area of parallelogram $= \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc.$

$$\det \begin{pmatrix} c & a \\ d & b \end{pmatrix} = bc - ad = - \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\text{Area} = \left| \det ( \quad ) \right|$$

$\mathbb{R}^3$



$$\text{Vol} = \left| \det(\vec{v}_1, \vec{v}_2, \vec{v}_3) \right|$$

# Determinants as Area, or Volume

**Theorem**

The volume of the parallelpiped spanned by the columns of an $n \times n$ matrix $A$ is $|\det A|$.

**Key Geometric Fact (which works in any dimension).** The area of the parallelogram spanned by two vectors $\vec{a}, \vec{b}$ is equal to the area spanned by $\vec{a}, c\vec{a} + \vec{b}$, for any scalar $c$.

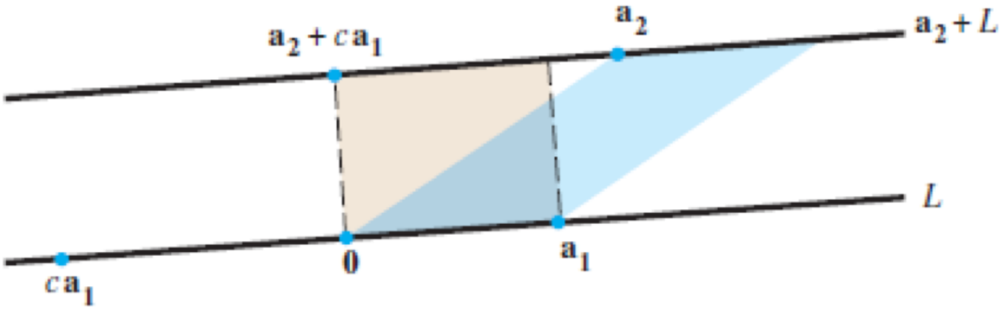
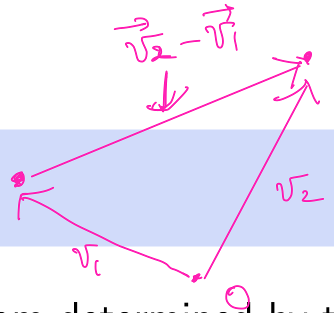


**FIGURE 2** Two parallelograms of equal area.

$$\det(a_1, a_2) = \det(a_1, a_2 + c \cdot a_1)$$

Replacement



# Example 1

Calculate the area of the parallelogram determined by the points $(-2,-2), (0,3), (4,-1), (6,4)$

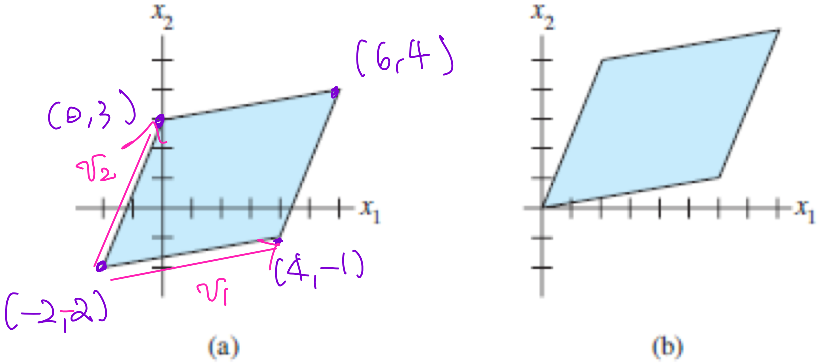


**FIGURE 5** Translating a parallelogram does not change its area.

$$\vec{v_1} = \begin{pmatrix}4\\-1\end{pmatrix} - \begin{pmatrix}-2\\-2\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix}$$

$$\vec{v_2} = \begin{pmatrix}0\\3\end{pmatrix} - \begin{pmatrix}-2\\-2\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix}$$

$$\text{Area} = \left|\det\left(\vec{v_1} \quad \vec{v_2}\right)\right| = \left|\det\begin{pmatrix}6 & 2\\1 & 5\end{pmatrix}\right| = |30-2| = \underline{\underline{28}}$$

$$\underline{\det(A \cdot B) = \det(A) \cdot \det(B)}$$

# Linear Transformations

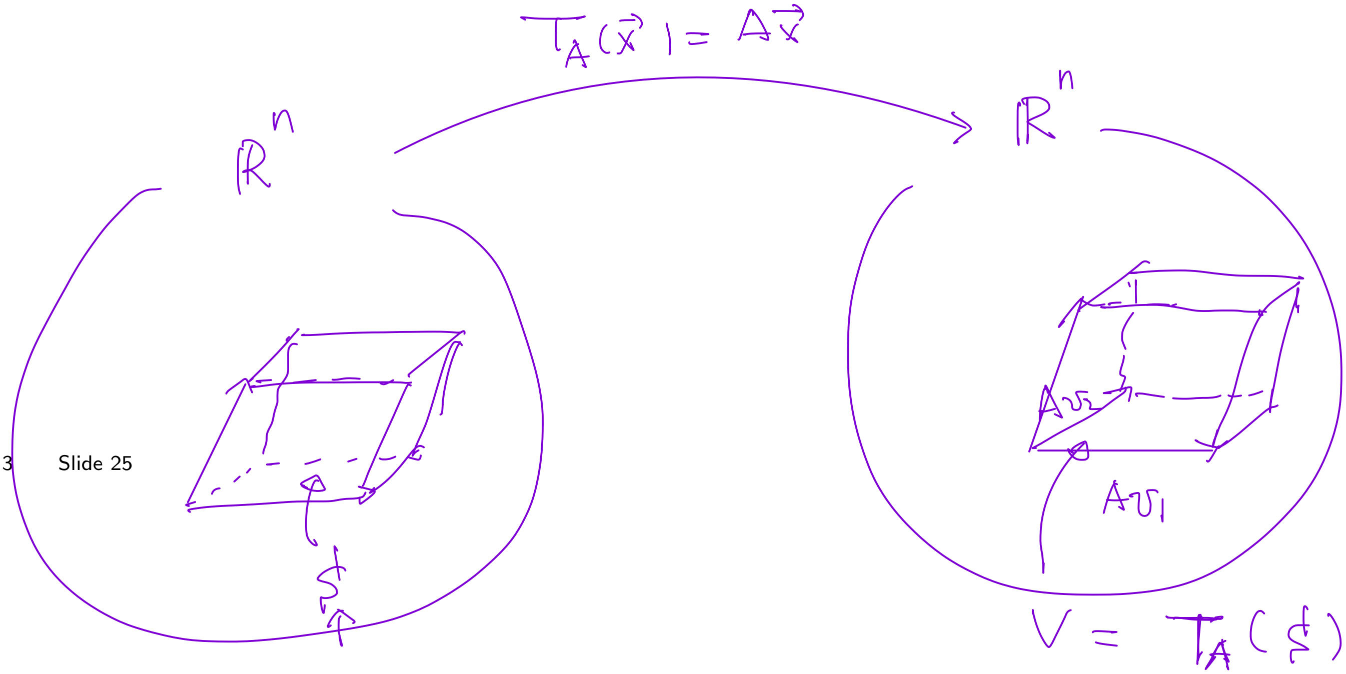
**Theorem**

If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$, and $S$ is some parallelogram in $\mathbb{R}^n$, then

$$\text{volume}\,(T_A(S)) = |\det(A)| \cdot \text{volume}(S)$$

An example that applies this theorem is given in this week's worksheets.

$A \in \mathbb{R}^{n \times n}$



$$V = \underline{T_A(S)}$$

defined by $v_1, v_2, \dots, v_n$ $\quad A \cdot [v_1 \cdots v_n]$ $\quad$ defined by $Av_1, \dots, Av_n$

$$\text{Vol}(T_A(S')) = \Big|\det(\overset{\shortparallel}{Av_1, \dots, Av_n})\Big| = |\det(A)| \cdot \underbrace{|\det[v_1 \cdots v_n]|}_{\overset{\shortparallel}{\text{Vol}(S')}}$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det[\,u_1 + v_1 ,\ v_2 ,\ v_3 ,\ \cdots ,\ v_n\,]$$

$$= \det[\,u_1 ,\ v_2 ,\ \cdots ,\ v_n\,] + \det[\,v_1 ,\ v_2 ,\ \cdots ,\ v_n\,]$$

↑

Exercize Cofactor Expansion.

# Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**
We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

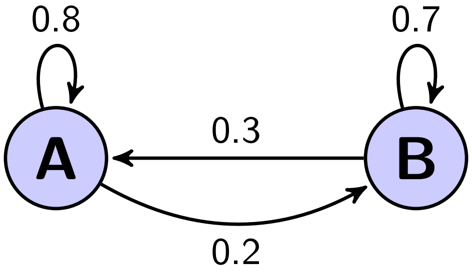
**Objectives**
For the topics covered in this section, students are expected to be able to do the following.

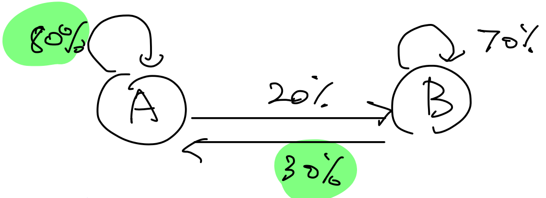
1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

# Example 1

- A small town has two libraries, $A$ and $B$.
- After 1 month, among the books checked out of $A$,
  - 80% returned to $A$
  - 20% returned to $B$
- After 1 month, among the books checked out of $B$,
  - 30% returned to $A$
  - 70% returned to $B$

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After $n$ months? A place to simulate this is `http://setosa.io/markov/index.html`





Initial State

books in A initially.

$$\vec{x}_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

in B

$\xrightarrow{\text{After 1 month}}$

A: 80% from A + 30% from B

$0.8 \times 0.5 + 0.3 \times 0.5 = 0.55$

B: 70% from B + 20% from A

$0.7 \times 0.5 + 0.2 \times 0.5 = 0.45$

# Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

$$\vec{x}_1 = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} \longrightarrow \vec{x}_2 = \begin{bmatrix} 0.8 \times 0.55 + 0.3 \times 0.45 \\ 0.7 \times 0.45 + 0.2 \times 0.55 \end{bmatrix}$$

After $k$ months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

$$\vec{x}_{k-1} = \begin{bmatrix} a \\ b \end{bmatrix} \longrightarrow \vec{x}_k = \begin{bmatrix} 0.8 \cdot a + 0.3 \cdot b \\ 0.2\, a + 0.7 \cdot b \end{bmatrix}$$

$$= \underset{\overset{\shortparallel}{P}}{\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{= \vec{x}_{k-1}}$$

$$= P \vec{x}_{k-1}$$

$$\vec{x}_k = P \vec{x}_{k-1} = \underbrace{P \cdot ( P \vec{x}_{k-2})} = P^2 \cdot \vec{x}_{k-2} = P^3 \vec{x}_{k-3} =$$

$$\ldots = P^k \cdot \vec{x}_0$$

# Markov Chains

A few definitions:

$$\vec{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad a_1, \dots, a_n \geqslant 0 \qquad a_1 + \dots + a_n = 1$$

- A **probability vector** is a vector, $\vec{x}$, with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, $P$, whose columns are probability vectors.

  Prob. vectors

  $$P = [\vec{c}_1 \ \vec{c}_2 \ \cdots \ \vec{c}_n]$$
- A **Markov chain** is a sequence of probability vectors $\vec{x}_k$, and a stochastic matrix $P$, such that:

  $$\{\vec{x}_0, \vec{x}_1, \dots, \vec{x}_k, \dots\}$$

  $$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \dots \quad \Rightarrow \quad \vec{x}_k = P^k \cdot \vec{x}_0$$
- A **steady-state vector** for $P$ is a vector $\boxed{\vec{q}}$ such that $P\vec{q} = \vec{q}$.

  prob.

  $$\vec{q} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad a + b = 1 \qquad a, b \geqslant 0$$

Ex

$$P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

# Example 2

Determine a steady-state vector for the stochastic matrix

$$P = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix} \leftarrow \text{stochastic matrix.}$$

Find $\vec{q} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad a + b = 1 \quad a, b \geq 0$

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$P \cdot \vec{q} = \vec{q}$$

$$P \cdot \vec{q} - I \cdot \vec{q} = \vec{0}$$

$$(P - I) \cdot \vec{q} = \vec{0}$$

$$\begin{bmatrix} 0.8 - 1 & 0.3 \\ 0.2 & 0.7 - 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leftarrow \text{homogeneous system}$$

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \xrightarrow[R_1 \to -5R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$$

$$a = \frac{3}{2} \cdot b$$

$$1 = a + b = \frac{3}{2} \cdot b + b = \frac{5}{2} \cdot b \Rightarrow b = \frac{2}{5}, \quad a = \frac{3}{2} \cdot \frac{2}{5} = \frac{3}{5}$$

$$\vec{q} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

k large

|  | $x_0$ | $x_1$ | $x_2$ | $x_k$ |
|---|---|---|---|---|
| A | 1000 | 1100 | . | $\approx 1200$ |
| B | 1000 | 900 | . | $\approx 800$ |

$\vec{q} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$

## Convergence

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

as $k \to \infty$.

**Definition**: a stochastic matrix $P$ is **regular** if there is some $k$ such that $P^k$ only contains strictly positive entries.

**Theorem**

is good enough

If $P$ is a regular stochastic matrix, then $P$ has a unique steady-state vector $\vec{q}$, and ② $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

Ex

$P = \begin{bmatrix} 0.7 & 0 \\ 0.3 & 1 \end{bmatrix}$ is not regular

$P^2 = \begin{bmatrix} 0.7 & 0 \\ 0.3 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0 \\ 0.3 & 1 \end{bmatrix}$

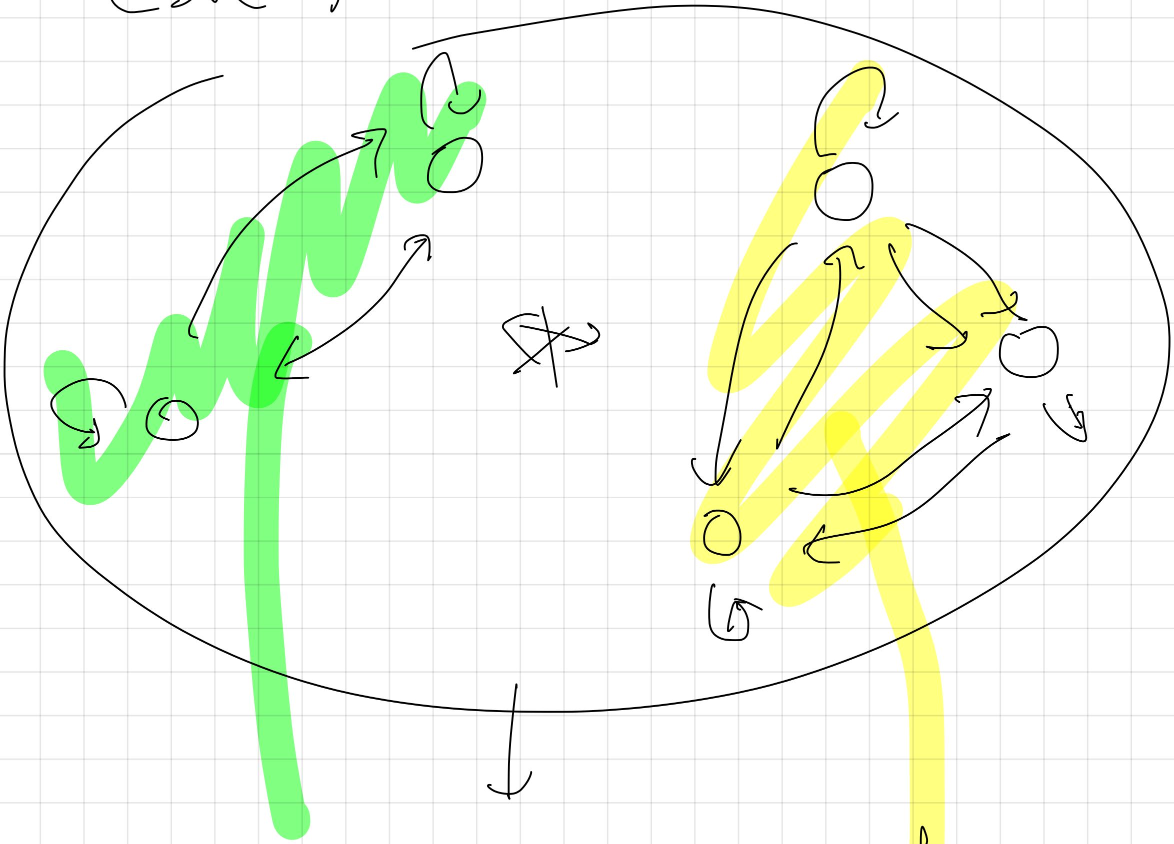
$= \begin{bmatrix} (0.7)^2 & 0 \\ (0.3)(0.7)+0.3 & 1^2 \end{bmatrix}$

$P^3 = \begin{bmatrix} (0.7)^3 & 0 \\ * & 1^3 \end{bmatrix} \quad \ldots \quad P^k = \begin{pmatrix} (0.7)^k & 0 \\ * & 1^k \end{pmatrix}$

Ex $P = \begin{bmatrix} 0.7 & 1 \\ 0.3 & 0 \end{bmatrix} \longrightarrow$ Regular.

$$P^2 = \begin{bmatrix} 0.7 & 1 \\ 0.3 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 1 \\ 0.3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (0.7)^2 + 0.3 & 0.7 \\ (0.3)(0.7) & 0.3 \end{bmatrix}$$



$$\left[\begin{array}{c|c} \begin{bmatrix} 2\times 2 \end{bmatrix} & \underline{0} \\ \hline \underline{0} & \begin{bmatrix} 3\times 3 \end{bmatrix} \end{array}\right]^k$$

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$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ prob. vector $x_1, \dots, x_n \geq 0$, $x_1 + \cdots + x_n = 1$

$P = [\, \vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n \,]$ stochastic matrix if $\vec{a}_1, \dots, \vec{a}_n$ are prob. vectors

$\{\vec{x}_0, \dots, \vec{x}_k, \dots\}$ MC if $\vec{x}_{k+1} = P \cdot \vec{x}_k$

$(P\vec{q} = \vec{q})$

Thm: If $P$ is regular, there is unique steady state $\vec{q}$

$\vec{x}_k \to \vec{q}$

regular ↑ $P^k$ has no zero entries for some $k$

# Example 3

A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

| | | rented from | | |
|---|---|---|---|---|
| | | A | B | C |
| | A | .8 | .1 | .2 |
| returned to | B | .2 | .6 | .3 |
| | C | .0 | .3 | .5 |

There are 10 cars at each location today.

a) Construct a stochastic matrix, $P$, for this problem.

b) What happens to the distribution of cars after a long time? You may assume that $P$ is regular.

$\vec{x}_k \rightsquigarrow \boxed{\vec{q}}$ when $k$ is large

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0.8 & 0.1 & 0.2 \\ B & 0.2 & 0.6 & 0.3 \\ C & 0 & 0.3 & 0.5 \end{array}$$

$$P^2 = \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.2 & 0.6 & 0.3 \\ 0 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.2 & 0.6 & 0.3 \\ 0 & 0.3 & 0.5 \end{bmatrix}$$

has no zero entries.

$P$ is regular.

$P\vec{q} = \vec{q}$

$(P - I)\vec{q} = \vec{0}$

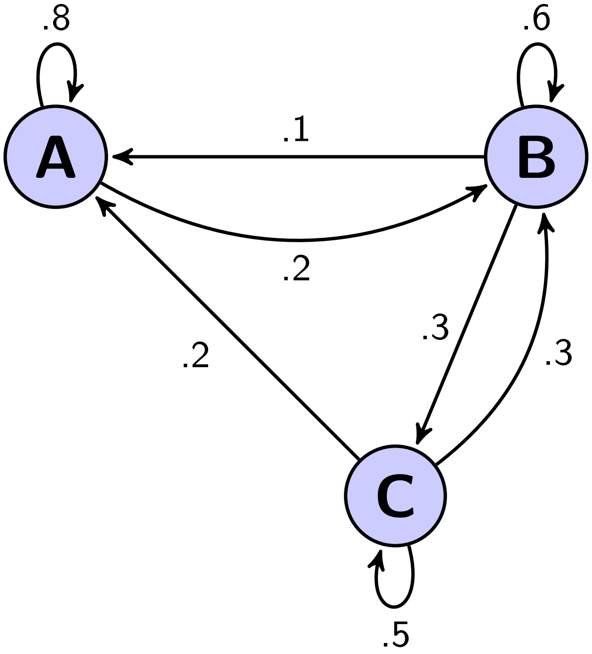
$$\begin{bmatrix} \overset{-1}{0.8} & 0.1 & 0.2 \\ 0.2 & \overset{-1}{0.6} & 0.3 \\ 0 & 0.3 & \overset{-1}{0.5} \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 1 & 2 \\ 2 & -4 & 3 \\ 0 & 3 & -5 \end{bmatrix} \xrightarrow[R_3 \to R_2 + R_3]{R_2 \to R_1 + R_2} \begin{bmatrix} -2 & 1 & 2 \\ 0 & -3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} -2 & 0 & 11/3 \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{11}{6} \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} a - \frac{11}{6}c = 0 \\ b - \frac{5}{3}c = 0 \end{matrix}$$

$$\vec{q} = t \underline{\begin{bmatrix} 11/6 \\ 5/3 \\ 1 \end{bmatrix}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \left(\frac{11}{6} + \frac{5}{3} + 1\right) \cdot t = 1$$

$$t = \frac{6}{27}$$

$$\vec{q} = \begin{bmatrix} 11/27 \\ 10/27 \\ 6/27 \end{bmatrix}$$



$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

# Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

$$A\vec{v} = \lambda\vec{v}$$

$$Av - \lambda I\vec{v} = \vec{0}$$

If $(A - \lambda I)\vec{v} = \vec{0}$ has nontrivial solution,
homogeneous eqn.

$\vec{v}$ : eigenvector
$\lambda$ : eigenvalue

$\lambda$ : lambda

# Eigenvectors and Eigenvalues

If $A \in \mathbb{R}^{n \times n}$, and there is a ① $\vec{v} \neq \vec{0}$ in $\mathbb{R}^n$, and

② $$A\vec{v} = \boxed{\lambda}\vec{v}$$

the set of complex numbers

then $\vec{v}$ is an **eigenvector** for $A$, and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

Note that

- We will only consider square matrices.
- If $\lambda \in \mathbb{R}$, then
  - when $\lambda > 0$, $A\vec{v}$ and $\vec{v}$ point in the same direction
  - when $\lambda < 0$, $A\vec{v}$ and $\vec{v}$ point in opposite directions
- Even when all entries of $A$ and $\vec{v}$ are real, $\lambda$ can be complex (a rotation of the plane has no **real** eigenvalues.)
- We explore complex eigenvalues in Section 5.5.

$$A\vec{v} = \lambda\vec{v}$$

# Example 1

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A \cdot \vec{v}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \overset{\lambda}{\overset{\shortparallel}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b) $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \overset{\lambda}{\overset{\shortparallel}{0}} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

c) $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

No. $\vec{v}_3 = \vec{0}$ cannot be eigenvector

$A\vec{v} = \lambda\vec{v}$

$(A-3I)\vec{v} = \vec{0}$ Find nonzero $\vec{v}$.

# Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

Need: $(A-3I)\vec{v} = 0$ has nontrivial Sol.

$\Leftrightarrow$ $(A-3I)$ is NOT invertible

$\Leftrightarrow$ $\det(A-3I) = 0$

$$\det(A-3I) = \det\begin{bmatrix} 2-3 & -4 \\ -1 & -1-3 \end{bmatrix} = \det\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix}$$

$$= (-1)\cdot(-4) - (-1)\cdot(-4) = 0.$$

$\underline{E_\lambda}$ = Eigenspace for $\lambda$ = { Solution of $(A-\lambda I)\vec{v}=0$ } = $\boxed{\text{Null}(A-\lambda I)}$

= { $\underline{\text{Eigenvectors}}$ } $\cup$ { $\vec{0}$ }

# Eigenspace

**Definition**

Suppose $A \in \mathbb{R}^{n\times n}$. The eigenvectors for a given $\lambda$ span a subspace of $\mathbb{R}^n$ called the $\lambda$-**eigenspace** of $A$.

**Note:** the $\lambda$-eigenspace for matrix $A$ is $\mathrm{Nul}(A-\lambda I)$.

## Example 3

Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$A = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \quad \lambda = -1, \textcircled{2}$$

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ b \end{bmatrix}$$

$a-2b=0$

$\to a=2b$

Eigenspace for $\lambda=2$

$$E_2 = \text{Null}(A-2I) = \left\{ t\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t\in\mathbb{R} \right\}$$

Eigenvector

$$A-2I = \begin{bmatrix} 5-2 & -6 \\ 3 & -4-2 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 3 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$E_{-1} = \text{Null}(A+I) = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t\in\mathbb{R} \right\}$$

Eigenvector

# Theorems

Proofs for the most these theorems are in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.

$A - 0 \cdot I$

2. $A$ invertible $\Leftrightarrow$ $0$ is not an eigenvalue of $A$.

3. Stochastic matrices have an eigenvalue equal to 1.

4. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \lambda = 1, 2, 3$$

$$\det(A - 2I) = \begin{bmatrix} -1 & * & * \\ 0 & 0 & * \\ 0 & 0 & 1 \end{bmatrix} = -1 \cdot 0 \cdot 1 = 0 .$$

# Warning!

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

**Example**: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

- But the reduced echelon form of $A$ is:
- The reduced echelon form is triangular, and its eigenvalues are:

# Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

# Topics and Objectives

**Topics**

We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

**Objectives**

For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

# The Characteristic Polynomial

**Recall:**

$\lambda$ is an eigenvalue of $A \Leftrightarrow (A - \lambda I)$ is not Invertible

Therefore, to calculate the eigenvalues of $A$, we can solve

$$\det(A - \lambda I) = 0$$

The quantity $\det(A - \lambda I)$ is the **characteristic polynomial** of $A$.

The equation ~~quantity~~ $\det(A - \lambda I) = 0$ is the **characteristic equation** of $A$.

The roots of the characteristic polynomial are the eigenvalues of $A$.

10/6/23

$A \in \mathbb{R}^{n \times n}$

$\vec{v} \neq \vec{0}$ is an eigenvector of A if $A\vec{v} = \lambda \vec{v}$

$\lambda$ the corresponding eigenvalue.

How to find eigenvalues?

$A\vec{v} = \lambda \vec{v}$ has nontrivial solutions

$(A - \lambda I)\vec{v} = \vec{0}$

$\Leftrightarrow$ $A - \lambda I$ is NOT invertible

$\Leftrightarrow$ $\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial}} = 0$ ← char. eqn.

How to find eigenvectors?

$E_\lambda = \text{Null}(A - \lambda I) =$ Solution sets of $\begin{cases} (A-\lambda I)\vec{v} = 0 & \text{or} \\ A\vec{v} = \lambda \vec{v} \end{cases}$

Eigenspace for $\lambda$ $= \{ \underline{\text{Eigenvectors}} \} \cup \{ \vec{0} \}$

nontrivial

# Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

$$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - 2\cdot 2$$

So the eigenvalues of $A$ are:

$$= 5\cdot 1 - 5\lambda - \lambda + \lambda^2 - 4$$

$$= \underbrace{\lambda^2 - 6\lambda + 1 = 0}$$

$$\underbrace{\lambda^2 - 6\lambda + 9}\ \boxed{+1} = 9$$

$$(\lambda - 3)^2 = 8$$

$$\lambda - 3 = \pm\sqrt{8}$$

$$\lambda = 3 \pm 2\sqrt{2}.$$ ← Eigenvalues.

# Characteristic Polynomial of $2 \times 2$ Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when $M$ is singular?

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - b \cdot c$$

$$= ad - a\lambda - d\lambda + \underline{\lambda^2} - bc$$

$$= \lambda^2 - \underbrace{(a+d)}_{\substack{\shortparallel \\ \text{trace of } M \\ \shortparallel \\ tr(M)}}\lambda + \underbrace{(ad-bc)}_{\substack{\shortparallel \\ \det(M)}}$$

If $M$ is singular, $\det(M) = 0$.

$$\phi(\lambda) = \det(M - \lambda I) = \lambda^2 - (a+d)\lambda$$

$$= \lambda \cdot (\lambda - (a+d)) = 0$$

$\lambda = 0$ or $a+d = tr(M)$.

Ex $(\lambda-1)^{1} \cdot \underset{\sim\sim\sim}{(\lambda-2)}^{\textcircled{2}} \underline{(\lambda-4)}^{\textcircled{5}} = 0$ has alg. multi. = 2

$\lambda = 1, 2, 4$ has alg. multi. = 5.

has alg. multi. = 1

# Algebraic Multiplicity

**Definition**

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

**Example**

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & & & 0 \\ & -\lambda & & \\ & & -1-\lambda & \\ 0 & & & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda)(-1-\lambda)(-\lambda)$$

$$= (\lambda-1)^{\textcircled{1}} \lambda^2 \underset{\sim\sim\sim}{(\lambda+1)}^{\textcircled{1}}$$

$$\lambda = 1, \quad \underset{\sim\sim}{\textcircled{-1}}, \quad 0$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$1 + 1 + 2 = \underline{\underline{4}}$ ← size of matrix.

$1 \leq$ Geom. Multi. $\leq$ Alg. Multi.

## Geometric Multiplicity

**Definition**

The **geometric multiplicity** of an eigenvalue $\lambda$ is the dimension of $\mathrm{Null}(A - \lambda I)$.

1. Geometric multiplicity $\geq 1$ is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

free var.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^{\textcircled{2}} = 0$$

$\lambda = 0$ has alg. multi 2

Geom Multi. = dim ( Null ( A − 0·I ) )

"

dim of Eigenspace = # of free var. = 1.

"

maximum # of lin. indep. eigenvectors.

# Example

Give an example of a $4 \times 4$ matrix with $\lambda = 0$ the only eigenvalue, but the geometric multiplicity of $\lambda = 0$ is one.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = 0$$

# Recall: Long-Term Behavior of Markov Chains

**Recall**:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

  as $k \to \infty$.
- If $P$ is regular, then there is a steady state vector $\vec{q}$

$\vec{x}_k \to \vec{q}$ as $k \to \infty$

$P\vec{q} = 1\vec{q}$

$\vec{q}$ is an eigenvector $\lambda = 1$

**Now lets ask**:

- If we don't know whether $P$ is regular, what else might we do to describe the long-term behavior of the system?
- What can eigenvalues tell us about the behavior of these systems?

Fact $P$ : stochastic $\Rightarrow$ $\lambda = 1$ eigenvalue.

$P - I$ Not invertible

$$= \begin{bmatrix} -1 & \cdot & \cdot & \\ \cdot & -1 & \cdot & \\ \vdots & \vdots & -1 & \\ \cdot & \cdot & \cdot & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$$\in \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_1 + \cdots + x_n = 0 \right\}$$

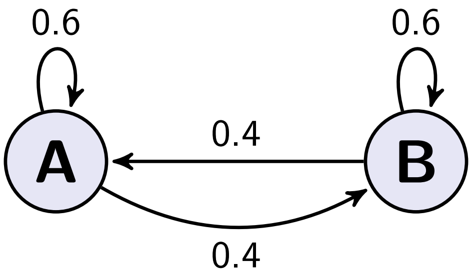
# Example: Eigenvalues and Markov Chains

Consider the Markov Chain:

a lin. combi. of $\vec{v}_1, \vec{v}_2$

$$\vec{x}_{k+1} = P\vec{x}_k = \overset{=P}{\begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix}} \vec{x}_k, \quad k = 0, 1, 2, 3, \dots, \quad \vec{x}_0 \overset{||}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:



**Goal**: use eigenvalues to describe the long-term behavior of our system.

$$\det(P - \lambda I) = \lambda^2 - (\underbrace{\overset{a+d}{1.2}}) \lambda + (\overset{ad-bc}{0.6^2 - 0.4^2})$$

$$= \lambda^2 - 1.2\lambda + 0.2 = 0$$

$$5\lambda^2 - 6\lambda + 1 = 0$$

$$(\lambda - 1)(5\lambda - 1) = 0 \qquad \lambda = 1, \ \frac{1}{5}$$

$$\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1/2 \\ -\frac{1}{2} \end{bmatrix}$$

Null $(A - I)$

Null $(A - \frac{1}{5} I)$

$$\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_1 + \vec{v}_2$$

$$A\vec{x}_0 = \underline{A\vec{v}_1} + \underline{A\vec{v}_2} = \boxed{\vec{v}_1} + \underline{\frac{1}{5}} \cdot \vec{v}_2 = \vec{x}_1$$

$$A\vec{x}_1 = \vec{v}_1 + \left(\frac{1}{5}\right)^2 \vec{v}_2 = \vec{x}_2$$

$$\vec{x}_k = \underset{\uparrow}{\boxed{\vec{v}_1}} + \underbrace{\left(\frac{1}{5}\right)^k \vec{v}_2}_{\downarrow\, 0}$$

limit of $\vec{x}_k$

What are the eigenvalues of $P$?

What are the corresponding eigenvectors of $P$?

Use the eigenvalues and eigenvectors of $P$ to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

# Similar Matrices

**Definition**

Two $n \times n$ matrices $A$ and $B$ are **similar** if there is a matrix $P$ so that $A = PBP^{-1}$.

**Theorem**

If $A$ and $B$ similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section 5.2, but doesn't have exercises on this concept until 5.3.
- Two matrices, $A$ and $B$, do not need to be similar to have the same eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

# Additional Examples (if time permits)

1. True or false.
    a) If $A$ is similar to the identity matrix, then $A$ is equal to the identity matrix.
    b) A row replacement operation on a matrix does not change its eigenvalues.
2. For what values of $k$ does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$