Practice Problems for Exam 2

1. Let X and Y be discrete random variables with joint PMF $p(x, y)$ given by

$$p(0,0) = p(0,1) = p(1,1) = p(1,2) = \frac{1}{4}.$$

   (a) Find the conditional expectation $\mathbb{E}[X|Y = 1]$.

   (b) Find the covariance $\text{Cov}(X, Y)$.

   (c) Are they independent?

2. Let X, Y be continuous random variables with joint PDF

$$f(x, y) = \begin{cases} Cxy, & 0 \le x \le 1, 0 \le y \le 1, y \ge x \\ 0 & \text{otherwise} \end{cases}$$

   for some constant $C > 0$.

   (a) Find the constant C.

   (b) Find the marginal PDF of Y.

   (c) Find the conditional probability $\mathbb{P}(Y \ge 3/4|X = 1/2)$.

3. Let X be a continuous random variable with CDF

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{1}{2}x^2 & 0 \le x < 1, \\ \frac{1}{2}x & 1 \le x < 2, \\ 1 & x \ge 2. \end{cases}$$

   (a) Find $\mathbb{P}(0.5 \le X \le 1.5)$.

   (b) Find the PDF of X.

4. Let $\Phi(x)$ be the CDF of the standard normal distribution.

   (a) Let $X \sim N(3, 16)$. Write down $\mathbb{P}(1 \le X < 4)$ in terms of $\Phi(x)$ for $x \ge 0$. Then use the corresponding table to compute the probability.

   (b) Let X, Y be independent normal random variables with common distribution $N(3, 16)$. Use the corresponding table to compute the probability $\mathbb{P}(1 \le X < 2, 4 < Y \le 5)$.

5. Let $X \sim \text{Exp}(2)$.

   (a) For $0 < p < 1$, find the $(100p)$-th percentile of X.

   (b) Let $h(x)$ be a function defined by

$$h(x) = \begin{cases} 1, & x \le 1 \\ x, & x > 1. \end{cases}$$

   Compute the expectation $\mathbb{E}[h(X)]$.

   (c) Compute the expectation $\mathbb{E}[\max\{X, X^2\}]$.

6. Roll two 4-sided dice. Let X be the outcome of the first die and Y be the sum of two outcomes.

   (a) Find the conditional expectation $\mathbb{E}[X|Y = 4]$.

   (b) Find the covariance $\text{Cov}(X, Y)$.

   (c) Are they independent?

7. In a bowl, there are 3 balls numbered from 1 to 3. You extract two of them without replacement. Let X_1 be the result of the first extraction and X_2 the result of the second one.

   (a) Compute the joint probability mass function $p(x_1, x_2)$ of X_1 and X_2.

   (b) Compute the probability mass function of $Y = X_1 + X_2$ and the expected value $\mathbb{E}[X_1 + X_2]$.

   (c) Compute the marginal $p_{X1}(x_1)$ with respect to X_1 of $p(x_1, x_2)$ and the conditional probability mass function $p(X_1|X_2)(x_1|x_2)$.

   (d) Compute $\text{Cov}(X_1, X_2)$. Are they independent? Why?

8. Let X be an exponential RV with parameter $\lambda = 2$ and Y be a uniform RV between 0 and 1. Assume that X and Y are independent.

   (a) Compute the joint probability distribution function $f(x, y)$ of X and Y.

   (b) Compute the probability that $X > 3$ and $Y > 0.5$.

   (c) Compute the probability that X is larger than Y.

9. Let X_1 and X_2 be two independent normal standard RVs. Let

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 - X_2$$

   (a) Compute the expected value and variance of Y_1 and Y_2.

   (b) Compute $\text{Cov}(Y_1, Y_2)$ and $\text{Corr}(Y_1, Y_2)$.

   (c) Can Y_1 and Y_2 be independent?

10. A passenger can take two different bus lines to get to work. Both lines stop at the same bus stop. Let T_1 be the RV that gives the time of arrival of the bus of line 1 and T_2 the RV giving the time of arrival of the bus of line 2. You know that T_1 is exponential with parameter 10 and T_2 is exponential with parameter 20 when time is measured in minutes. Let T be the random variable that gives the time our passenger will wait for a bus, i.e., $T = \min(T_1, T_2)$ the smallest between T_1 and T_2.

   (a) Compute $\mathbb{P}(T > t)$.

   (b) What is the PDF of T?

   (c) What is the average time the passenger will wait at the bus stop?

11. In a room, you have 10 independent bulbs each of which has a life time distributed exponentially with parameter 0.1 when the time is measured in days. What is the probability distribution of the number of working bulbs after 10 days?

12. A person works to install the operating system on a computer and another worker configures it after installation. Let X be the time needed by the first worker, then we know that it is distributed exponentially with parameter 1 when the time is measured in hours. If the first worker needed x hours to complete his job, we know that the second worker will need a time Y that is exponentially distributed with the parameter $0.5x$.

   (a) Write the joint PDF of X and Y.

   (b) Write an expression for the expected value of the total time needed to complete the installation and configuration.

13. The PDF of Y is given by $f_Y(y) = \frac{c}{y^3}$ for $1 < y < \infty$ and otherwise 0.

   (a) Find the value of c so that f_Y is a PDF.

   (b) Compute $\mathbb{E}[Y]$.

14. Let X be a continuous random variable with the PDF

$$f_X(x) = \begin{cases} x + 1, & -1 < x < 0, \\ 1 - x, & 0 \le x < 1. \end{cases}$$

   Find the CDF of X. Draw the graph of the CDF.

15. Let X be the total amount of medical claims (in \$ 100,000) of the employees of a company. Assume that the PDF $f(x) = 30x(1 - x)^4$ for $0 < x < 1$.

   (a) Find $\mathbb{E}[X]$ and $\text{Var}(X)$.

(b) Find the probability that the total exceeds \$20,000.

16. Let X be the number of alpha particle emissions of carbon-14 that are counted by a Geiger counter each second. Assume that the distribution of X is Poisson with mean 5. Let W be the time in seconds before the third count is made.

    (a) What is the distribution of W?

    (b) Find $\mathbb{P}(W \le 1)$.

17. A loss (in \$ 100,000) due to fire in a building has a PDF $f(x) = \frac{1}{6}e^{-x/6}$, $0 < x < \infty$. Find the conditional probability that the loss is greater than 8 given that it is greater than 5.

18. Let $X \sim N(650, 400)$.

    (a) Find $\mathbb{P}(600 \le X < 660)$.

    (b) Find a constant $c > 0$ such that $\mathbb{P}(|X - 650| \le c) = 0.95$.

19. Let $X \sim N(0, 4)$ and $W = X^2$. Find the PDF of W.

20. Let X be a random variable with the CDF

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{4}, & 0 \le x < 1, \\ \frac{x+1}{4}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$

    Find $\mathbb{E}[X], \mathrm{Var}(X), \mathbb{P}(\frac{1}{4} < X < 1), \mathbb{P}(X = 1), \mathbb{P}(X = \frac{1}{2}), \mathbb{P}(\frac{1}{2} \le X < 2)$.

21. A loss X on a car has an exponential distribution with a mean of \$5000. If the loss X on a car is greater than the deductible of \$500, the difference $X - 500$ is paid to the owner of the car. Considering zero (if $X < 500$) as a possible payment, find the expectation of the payment.

22. Let X and Y be discrete random variables with joint PMF

$$f(x, y) = \frac{x + y}{21}, \qquad x = 1, 2, \quad y = 1, 2, 3.$$

    (a) Find the marginal PMFs of X and Y.

    (b) Find $\mathbb{P}(X + Y \le 3)$.

23. Let X and Y be discrete random variables with joint PMF

$$f(0, 0) = f(1, 2) = 0.2, \quad f(0, 1) = f(1, 1) = 0.3.$$

    (a) Find $\mathrm{Cov}(X, Y)$.

    (b) Find the least square regression line.

24. Let X and Y be discrete random variables with joint PMF $f(x, y) = \frac{1}{9}$ for $(x, y) \in S$ where

$$S = \{(x, y) : 0 \le x \le 2, x \le y \le x + 2, x, y \text{ are integers}\}.$$

    Find $f_X(x), f_{Y|X}(y|1), \mathbb{E}[Y|X = 1]$, and $f_Y(y)$.

25. Suppose that X has a geometric distribution with parameter p. and suppose the conditional distribution of Y, given $X = x$, is Poisson with mean x. Find $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$.

26. Let X and Y have the joint PDF

$$f(x, y) = \frac{4}{3}, \qquad 0 < x < 1, \quad x^3 < y < 1$$

    and 0 otherwise. Find $\mathbb{P}(X > Y)$.

27. Let X be a uniform random variable over $(0, 2)$ and Y given $X = x$ be $U(0, x^2)$.

    (a) Find the joint PDF $f(x, y)$ and marginal $f_Y(y)$.

    (b) Find $\mathbb{E}[Y|X]$ and $\mathbb{E}[X|Y]$.

28. A certain type of electrical motors is defective with probability 1/100. Pick 1000 motors and let X be the number of defective ones among these 1000 motors. Using a normal approximation, (with/without) mid-point correction, write down an expression for the probability that among the 1000 motors 13 or less are defective.