Normal and Exponential Random Variables (Sec 5.4-7)

University of Illinois at Urbana–Champaign

Math 461 Spring 2022

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Normal random variables

X is a normal random variable with parameters $\underset{\text{mean}}{\mu}$ and $\underset{\text{variance}}{\sigma^2}$ if its density is

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

for $x \in \mathbb{R}$ and denoted by $\boxed{X \sim N(\mu, \sigma^2)}$.

If $\mu = 0$ and $\sigma = 1$, then we call X the standard normal random variable.

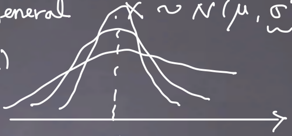
or Z

$$X \sim N(\mu, \sigma^2) \Rightarrow \begin{cases} X = \sigma Z + \mu, & Z \sim N(0,1) \\ Z = \dfrac{X-\mu}{\sigma} \sim N(0,1) \end{cases}$$

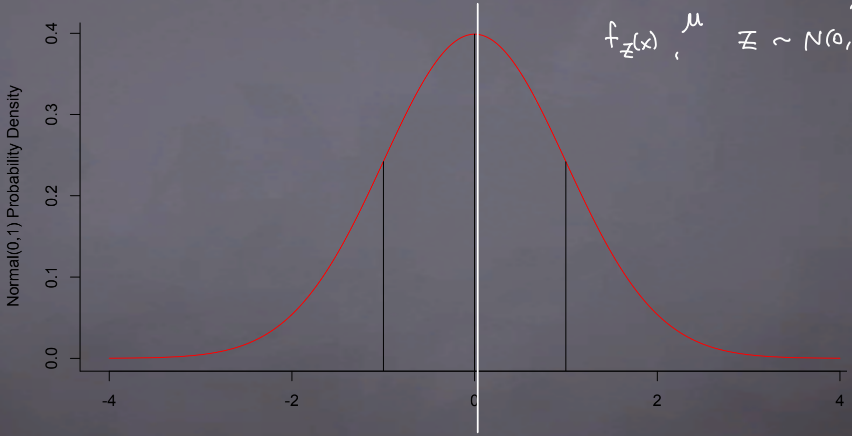
Probability density function

For general $X \sim N(\mu, \sigma^2)$

$f_X(x)$



$f_Z(x)$, $Z \sim N(0,1)$



$x = 0 = \mu$

Cumulative distribution function

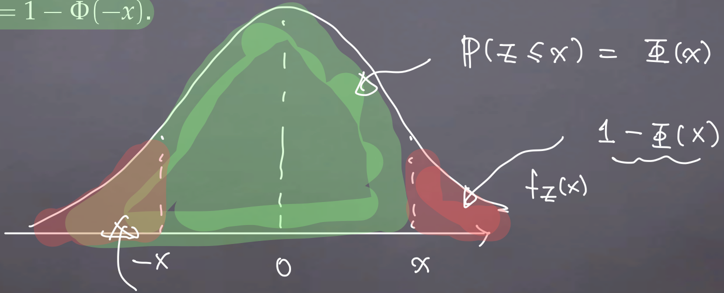
The cumulative distribution function of $N(0,1)$ is

$Z \sim N(0,1)$

$f_Z = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

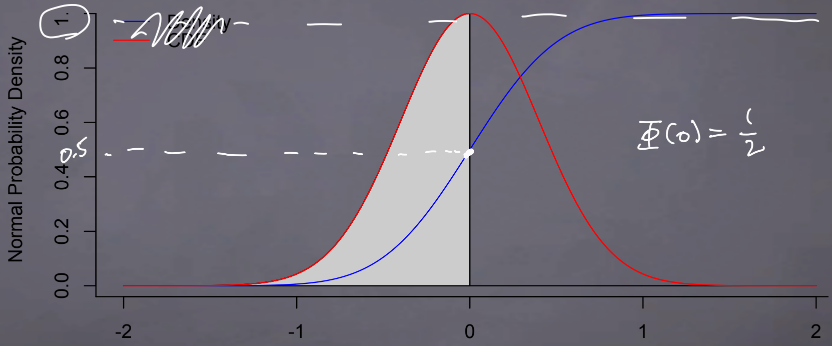
$$\mathbb{P}(Z \leqslant x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt.$$

Note that the cdf cannot be computed explicitly. Note also that $\Phi(x) = 1 - \Phi(-x)$.



$$\Phi(-x) = 1 - \Phi(x) \quad \text{or} \quad \Phi(x) = 1 - \Phi(-x)$$

Cumulative distribution function



$$\Phi(-x) = 1 - \Phi(x)$$

Cumulative distribution function

$$Z \sim N(0,1)$$

$$X = \sigma Z + \mu$$

$$= 3Z + 3$$

$$X \sim N(3,9)$$

$$\mu = 3$$

$$\sigma^2 = 9 \Rightarrow \sigma = 3$$

$$P(X \leq 4) = P(3Z+3 \leq 4)$$

$$= P\left(Z \leq \frac{1}{3}\right)$$

$$= \Phi\left(\frac{1}{3}\right)$$

$$= \int_{-\infty}^{\frac{1}{3}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\approx \Phi(0.33)$$

$$\approx 0.6293$$

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$$P(X \leq 1)$$

$$= P(3Z+3 \leq 1)$$

$$= P\left(Z \leq -\frac{2}{3}\right)$$

$$= \Phi\left(-\frac{2}{3}\right)$$

$$= 1 - \Phi\left(\frac{2}{3}\right)$$

$$\Phi(0.67)$$

$$\approx 1 - 0.7486$$

If n large, p small, $np = \lambda$ $\Rightarrow$ $S_n \sim \mathrm{Pois}(\lambda)$ approx.

Normal approximations to binomial

Let $S_n \sim \mathrm{Bin}(n,p)$ be the number of successes in n independent Bernoulli trials.

Then, we have seen that $\mathbb{E}[S_n] = np$, $\mathrm{Var}(S_n) = np(1-p)$.

For large n,

For large n, S_n is almost normal

$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{\mathrm{Var}(S_n)}} = \frac{S_n - np}{\sqrt{np(1-p)}} \approx N(0,1).$$

$X \sim N(np, np(1-p))$

$X = \sigma Z + \mu$

The approximation is good for $np(1-p) \geqslant 10$.

Compared to Poisson approximation, the success probability p needs not to be small.

$$Z = \frac{X - \mu}{\sigma} = \frac{X - np}{\sqrt{np(1-p)}}$$

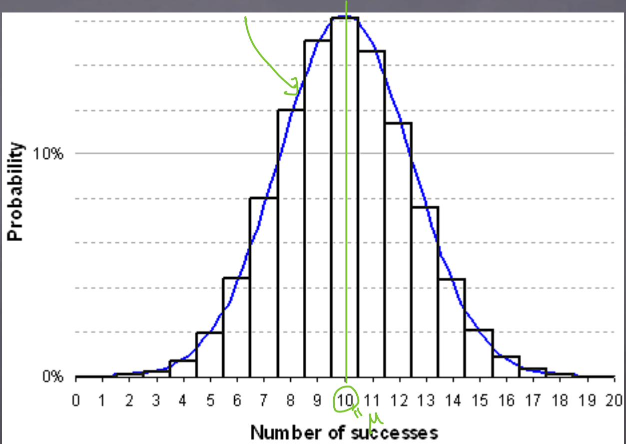
Normal approximations to binomial

$S \sim \mathrm{Bin}(20, \frac{1}{2})$

$\mu = 10 = 20 \cdot \frac{1}{2}$

$\sigma^2 = 20 \cdot \frac{1}{2} \cdot \frac{1}{2} = 5$

PDF of $X \sim N(10, 5)$.



Example

Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Normal approximation.

X = # of unacceptable items in 1500 items.

$\sim \mathrm{Bin}(150, 0.05)$

$E[X] = np = 1500 \cdot 0.05 = \frac{15}{2} = 7.5$

$Var(X) = np(1-p) = 1500 \cdot 0.05 \cdot 0.95$.

$$\frac{X - E[X]}{\sqrt{Var(X)}} = \frac{X - 7.5}{\sqrt{7.5 \cdot 0.95}} \approx Z \sim N(0,1)$$

$$\Phi\left(\frac{2.5}{\sqrt{Var}}\right).$$

$$P(X \leq 10) = P\left(\frac{X - 7.5}{\sqrt{\cdots}} \leq \frac{10 - 7.5}{\sqrt{\cdots}}\right) \approx P\left(Z \leq \frac{2.5}{\sqrt{Var}}\right)$$

Example

Each item produced by a certain manufacturer is, independently, of acceptable quality with probability .95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

Poisson

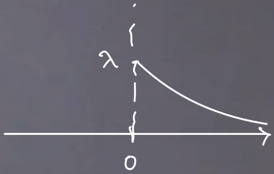
$n = 150$, $p = 0.05$ $\Rightarrow \lambda = np = 7.5$

$X \underset{\text{approx}}{\sim} \text{Pois}(7.5)$

Exponential random variable (Continuous Counterpart of Geometric RVs)

A random variable X is exponential with parameter $\lambda > 0$ if its density is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geqslant 0, \\ 0, & x < 0. \end{cases}$$



We denote by $X \sim \mathrm{Exp}(\lambda)$.

- "Lifetime".
- "Strong Relation to Poisson":

Expectation, Variance, CDF

Let $X \sim \mathrm{Exp}(\lambda)$ for $\lambda > 0$.

(i) The cumulative distribution function $F(x) = 1 - e^{-\lambda x}$.

(ii) $\mathbb{E}[X] = \frac{1}{\lambda}$.

(iii) $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

(i) $F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \lambda e^{-\lambda t}\,dt$

$= \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}$.

(ii) $\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^\infty x \lambda e^{-\lambda x}\,dx$ $\quad t = \lambda x$

$= \frac{1}{\lambda} \cdot \underbrace{\int_0^\infty t e^{-t}\,dt}_{=1} = \frac{1}{\lambda}$.

Memoryless property

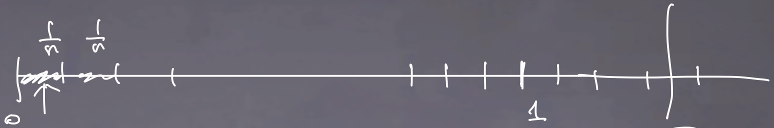
Let $s, t > 0$ and $X \sim \text{Exp}(\lambda)$ for $\lambda > 0$, then

$$\mathbb{P}(X > s + t | X > t) = \mathbb{P}(X > s).$$

Example

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles.

If a person desires to take a 5000-mile trip, what is the probability that the person will be able to complete the trip without having to replace the car battery?



$p = \frac{\lambda}{n}$ $\quad n$ sub intervals. $\quad t$

$$np = n \cdot \frac{\lambda}{n} = \lambda$$

$$Y \sim \text{Geom}\left(\frac{\lambda}{n}\right)$$

$$P(Y \le t) \xrightarrow{n \to \infty} \boxed{1 - e^{-\lambda t}} = P(X \le t)$$

$$X \sim \text{Exp}(\lambda).$$