Math 461: Probability Theory

Midterm 2, Spring 2022

Date: April 13, 2022

Solution

NAME:____________________ NetID:____________________

READ THE FOLLOWING INFORMATION.

- This is a 50-minute.
- This exam contains 8 pages (including this cover page) and 6 questions. Total of points is 60.
- Books, notes, and other aids are not allowed. Using calculator is okay.
- Show all steps to earn full credit.
- Do not unstaple pages. Loose pages will be ignored.

Question	Points	Score
1	10	
2	12	
3	10	
4	10	
5	8	
6	10	
Total:	60	

1. (10 points) Circle True or False. Do not justify your answer.

(a) True **False** If X has normal distribution with mean 7 and variance 10, then $Z = \frac{1}{10}(X - 7)$ has standard normal distribution.

$X \sim N(\mu, \sigma^2)$, $\mu = 7$, $\sigma = \sqrt{10}$

$Z = \frac{X-\mu}{\sigma} = \frac{X-7}{\sqrt{10}} \sim N(0,1)$

(b) **True** False If F is a CDF of a random variable of X, then $(F(x))^2$ is also a CDF of some other random variable.

If X, Y are independent RVs with the same $F(x)$, then the CDF of $\max\{X, Y\}$ is $F(x)^2$.

(c) **True** False The joint probability mass function of X and Y is given by

$$p(1,1) = p(2,1) = \frac{1}{8}, \quad p(1,2) = \frac{1}{4}, \quad p(2,2) = \frac{1}{2}.$$

Then, $\mathbb{E}[XY] = \frac{23}{8}$.

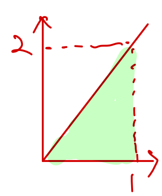
$\mathbb{E}[XY] = 1 \cdot 1 \cdot p(1,1) + 2 \cdot 1 \cdot p(2,1) + 1 \cdot 2 \cdot p(1,2) + 2 \cdot 2 \cdot p(2,2)$

$= \frac{1}{8}(1 + 2 + 4 + 16) = \frac{23}{8}$

(d) True **False** If $f(x, y)$ is the joint density of random variables X and Y given by

$$f(x,y) = \begin{cases} 1, & 0 < x < 1, 0 < y < 2x, \\ 0, & \text{otherwise}, \end{cases}$$

then X and Y are independent.



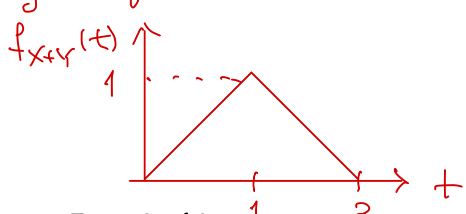
$f_X(x) = \int_0^{2x} 1\, dx = 2x$

$f_Y(y) = \int_{y/2}^{1} 1\, dx = 1 - \frac{y}{2}$

$f \neq f_X \cdot f_Y$

(e) True **False** If X, Y are independent uniform random variables over $(0, 1)$, then $X + Y$ is also a uniform random variable.

The density of $X + Y$ is



2. (12 points) Give definitions or explanations of the following.

(a) Write down the definition of Jointly continuous random variables X and Y.

There exists $f(x,y) \geqslant 0$ s.t. for all $A \subseteq \mathbb{R}^2$

$$\mathbb{P}((X,Y) \in A) = \iint_A f(x,y)\,dx\,dy.$$

(b) Let X, Y be jointly continuous random variables with joint density $f(x,y)$. What is the marginal densities of X and Y?

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

(c) Let X, Y be jointly continuous random variables with joint density $f(x,y)$. What is the conditional probability density functions of X given $Y = y$?

If $f_Y(y) > 0$, then $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

(d) Explain Normal approximation to binomial.

If $X \sim \text{Bin}(n,p)$ n : large, then

$\dfrac{X - np}{\sqrt{np(1-p)}}$ is approximately standard Normal.

3. A stick of length 1 is broken into two pieces at X, where X is a uniform random variable over $(0, 1)$.

(a) (5 points) Let Y be the length of the longer piece. Find a function $f : (0, 1) \to \mathbb{R}$ such that $Y = f(X)$.

Since $Y = \begin{cases} X & \text{if } X \geq \frac{1}{2} \\ 1-X & \text{if } 0 < X < \frac{1}{2} \end{cases}$,

$Y = f(X)$ where

$$f(x) = \begin{cases} x & \text{if } \frac{1}{2} \leq x < 1 \\ 1-x & \text{if } 0 < x < \frac{1}{2} \end{cases}.$$

(b) (5 points) Find $\mathbb{E}[Y]$.

$$E[Y] = E[f(X)] = \int_0^1 f(x)\,dx$$

$$= \int_0^{\frac{1}{2}} (1-x)\,dx + \int_{\frac{1}{2}}^1 x\,dx$$

$$= \frac{3}{4}.$$

4. Let X, Y be independent exponential random variables with parameters λ_1 and λ_2 respectively.

(a) (5 points) Is $Z := \min\{X, Y\}$ an exponential random variable? Justify your answer.

Yes.

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(\min\{X, Y\} \le t)$$
$$= 1 - \mathbb{P}(\min\{X, Y\} > t)$$
$$= 1 - \mathbb{P}(X > t, Y > t)$$
$$= 1 - \mathbb{P}(X > t)\,\mathbb{P}(Y > t)$$
$$= 1 - e^{-\lambda_1 t} \cdot e^{-\lambda_2 t}$$
$$= 1 - e^{-(\lambda_1 + \lambda_2)t} \qquad \therefore \quad Z \sim Exp(\lambda_1 + \lambda_2)$$

(b) (5 points) Is $W := \max\{X, Y\}$ an exponential random variable? Justify your answer.

No.

$$F_W(t) = \mathbb{P}(\max\{X, Y\} \le t)$$
$$= \mathbb{P}(X \le t)\,\mathbb{P}(Y \le t)$$
$$= (1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})$$
$$\ne 1 - e^{-\lambda t} \quad \text{for any } \lambda > 0.$$

5. (8 points) The life of a certain type of car battery X is normally distributed with mean 50 and the standard deviation 6 (in 1000 miles), that is, $X \in N(50, 6^2)$. Given that it has survived 50,000 miles, what is the conditional probability that the car battery survives another 15,000 miles?

$$P(X > 50+15 \mid X > 50) = \frac{P(Z > 2.5)}{P(Z > 0)}$$

$X = 6Z + 50$

$Z \sim N(0,1)$

$$= 2(1 - P(Z \leq 2.5))$$

$$= 2(1 - 0.9938)$$

$$= 0.0124$$

6. Let X, Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x^2y^2}, & x, y \geqslant 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let U, V be defined by

$$U = XY, \qquad V = X/Y.$$

(a) (10 points) Find the joint density $f_{U,V}(u,v)$ of U and V.

$$X = \sqrt{UV} = h_1(U,V)$$

$$Y = \sqrt{\frac{U}{V}} = h_2(U,V)$$

$$|J_h| = \left| \begin{pmatrix} \frac{1}{2}\sqrt{\frac{V}{U}} & \frac{1}{2}\sqrt{\frac{U}{V}} \\ \frac{1}{2}\frac{1}{\sqrt{UV}} & -\frac{1}{2}\sqrt{\frac{U}{V^3}} \end{pmatrix} \right| = \frac{1}{2V}$$

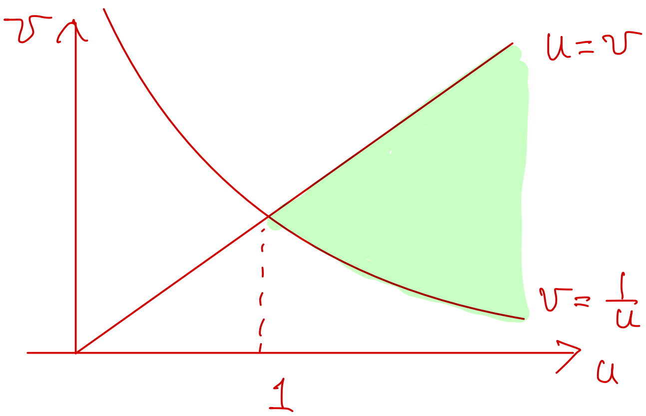
$$\Rightarrow f_{U,V}(u,v) = f_{X,Y}\left(\sqrt{uv}, \sqrt{\frac{u}{v}}\right) \cdot \frac{1}{2v}$$

$$= \begin{cases} \frac{1}{2u^2v} & \text{if } \sqrt{uv} \geqslant 1, \ \sqrt{\frac{u}{v}} \geqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b) (Extra credit: 4 points) Draw the region in the $u - v$ plane where $f_{U,V}(u,v) > 0$.

$$\sqrt{uv} \geq 1 \quad \text{and} \quad \sqrt{\frac{u}{v}} \geq 1$$

$$\Rightarrow \quad v \geq \frac{1}{u}, \quad u \geq v, \quad u \geq 1, \quad v > 0$$



(c) (Extra credit: 4 points) Find the marginal density $f_U(u)$ of U.

Method 1:

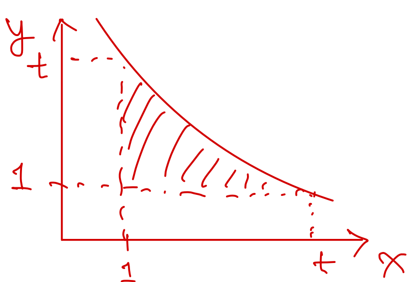
$$f_U(u) = \int_{\mathbb{R}} f_{U,V}(u,v)\,dv = \begin{cases} \int_{\frac{1}{u}}^{u} \frac{1}{2u^2 v}\,dv, & u \geq 1 \\ 0, & \text{o.w} \end{cases}$$

$$= \begin{cases} \frac{\log u}{u^2}, & u \geq 1 \\ 0, & \text{o.w.} \end{cases}$$

Method 2: For $t > 1$,

$$F_U(t) = P(XY \leq t) = \int_1^t \int_1^{\frac{t}{x}} \frac{1}{x^2 y^2}\,dy\,dx = \int_1^t \frac{1}{x^2} \cdot \left(1 - \frac{x}{t}\right) dx$$

$$= \left(1 - \frac{1}{t}\right) - \frac{1}{t} \log t$$



$$f_U(t) = F_U(t)' = \frac{1}{t^2} + \frac{\log t}{t^2} - \frac{1}{t^2}$$

$$= \frac{\log t}{t^2}$$