Homework 6 Solution

Math 461: Probability Theory, Spring 2022
Daesung Kim

Due date: Mar 11, 2022

1. An urn contains 4 white and 5 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

Solution: Let E be the event that a single drawing results in two white and two black balls. Then $\mathbb{P}(E) = \frac{\binom{4}{2}\binom{5}{2}}{\binom{9}{4}} = \frac{10}{21}$. Let X be the number of selections until E occurs. Thus X is a Geometric random variable with parameter $p = 10/21$. Then

$$\mathbb{P}(X = n) = \frac{11^{n-1} \cdot 10}{21^n}.$$

2. Let X be a negative binomial random variable with parameters r and p and let Y be a binomial random variable with parameters n and p. Show that

$$\mathbb{P}(X > n) = \mathbb{P}(Y < r).$$

Solution:

(a) **Probabilistic proof:** Consider a sequence of independent Bernoulli trials with success probability p. Then, Y is the number of success in the first n trials and X is the number of trials until r successes are obtained. If $X > n$, then it means that the number of success in the first n trials is less than r, that is, $Y < r$. Thus, we have $\{X > n\} = \{Y < r\}$.

(b) **Combinatoric proof:** Since

$$\mathbb{P}(X > n) = 1 - \mathbb{P}(X \leqslant n) = 1 - \sum_{k=r}^{n} \binom{k-1}{r-1} p^r (1-p)^{k-r} = 1 - \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \mathbb{P}(Y < r),$$

it suffices to show that

$$\sum_{k=r}^{n} \binom{k-1}{r-1} p^r (1-p)^{k-r} = \sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} \tag{1}$$

for all $r \leqslant n$. We fix $r > 0$ and prove the claim by induction on $n = r, r+1, \cdots$. If $n = r$, then

$$\sum_{k=r}^{r} \binom{k-1}{r-1} p^r (1-p)^{k-r} = p^r = \sum_{k=r}^{r} \binom{r}{k} p^k (1-p)^{r-k}.$$

Suppose (1) holds for $n \geqslant r$. Then,

$$\sum_{k=n+1}^{r}\binom{k-1}{r-1}p^r(1-p)^{k-r} = \binom{n}{r-1}p^r(1-p)^{n+1-r} + \sum_{k=n+1}^{r}\binom{k-1}{r-1}p^r(1-p)^{k-r}$$
$$= \binom{n}{r-1}p^r(1-p)^{n+1-r} + \sum_{k=r}^{n}\binom{n}{k}p^k(1-p)^{n-k}.$$

Using Pascal's identity ($\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$),

$$\sum_{k=r}^{n+1}\binom{n+1}{k}p^k(1-p)^{n+1-k} = \sum_{k=r}^{n+1}\binom{n}{k-1}p^k(1-p)^{n+1-k} + \sum_{k=r}^{n+1}\binom{n}{k}p^k(1-p)^{n+1-k}$$
$$= \binom{n}{r-1}p^r(1-p)^{n+1-r} + \sum_{k=r}^{n}\binom{n}{k}p^{k+1}(1-p)^{n-k} + \sum_{k=r}^{n}\binom{n}{k}p^k(1-p)^{n+1-k}$$
$$= \binom{n}{r-1}p^r(1-p)^{n+1-r} + \sum_{k=r}^{n}\binom{n}{k}p^k(1-p)^{n-k}.$$

Thus, (1) holds for $n+1$.

3. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x)^2 & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of c?
(b) What is the cumulative distribution function of X?

Solution: (a) We have $1 = \int_{-1}^{1} c(1-x)^2 dx = -\frac{c}{3}(1-x)^3|_{-1}^{1} = \frac{8}{3}c$, so that $c = \frac{3}{8}$.
(b) We have $\int_{-1}^{x} f(y)dy = -\frac{3}{8}(1-y)^3|_{-1}^{x} = \frac{3}{8}(8-(1-x)^3)$ if $-1 \leqslant x \leqslant 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ 8-(1-x)^3 & -1 \leqslant x < 1, \\ 1 & x \geqslant 1. \end{cases}$$

4. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2} & x \geqslant 0 \\ 0 & x \leqslant 0. \end{cases}$$

What is the probability that the system functions for at least 4 months?

Solution: Determine C: $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4$, so that $C = \frac{1}{4}$.
Now, we have $\mathbb{P}(X \geqslant 4) = \int_4^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}\Big|_4^\infty = 3e^{-2}$.

5. The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leqslant 10. \end{cases}$$

(a) Find $\mathbb{P}(X > 20)$.
(b) What is the cumulative distribution function of X?
(c) What is the probability that, of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

Solution:

(a) $\mathbb{P}(X > 20) = \int_{20}^{\infty} \frac{10}{x^2} dx = -\frac{10}{x}\Big|_{20}^{\infty} = \frac{1}{2}$.

(b)
$$F(x) = \begin{cases} 0 & x < 10 \\ 1 - \frac{10}{x} & x \geqslant 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is
$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

6. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function
$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$
what must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01?

Solution: We want to find C such that $F(C) \geqslant 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5\big|_0^C = 1 - (1-C)^5$. We want $1 - (1-C)^5 \geqslant 0.99$, i.e., $(1-C)^5 \leqslant 0.01$, hence $C \geqslant 1 - (0.01)^{0.2}$.

7. Compute $\mathbb{E}\, X$ if X has a density function given by

(a)
$$f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b)
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c)
$$f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & \text{otherwise.} \end{cases}$$

Solution:

(a) $\mathbb{E}\, X = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{4}\int_0^{\infty} x^2 e^{-\frac{x}{2}} dx = \frac{1}{4}\left(-2x^2 - 8x - 16\right) e^{-\frac{x}{2}}\big|_0^{\infty} = 4$.

(b) $\mathbb{E}\, X = \int_{-1}^{1} c(1-x^2) x dx = 0$ by symmetry.

(c) $\mathbb{E}\, X = \int_5^{\infty} x \frac{5}{x^2} dx = \int_5^{\infty} \frac{5}{x} dx = \infty$.

8. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.
(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?
(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

Solution: (a) Let X be uniform on $[0, 60]$ where X is the time in minutes after 7am when the passenger arrives at the station. Then

$$\begin{aligned}\mathbb{P}(\text{passenger goes to } A) &= \mathbb{P}(5 \leqslant X < 15) + \mathbb{P}(20 \leqslant X < 30) + \mathbb{P}(35 \leqslant X < 45) + \mathbb{P}(50 \leqslant X < 60) \\ &= \frac{2}{3}.\end{aligned}$$

(b) Same as above.

9. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.
(a) What is the probability that you will have to wait longer than 10 minutes?
(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

Solution: (a) $\mathbb{P}(X > 10) = \frac{2}{3}$
(b) $\mathbb{P}(X > 25 \mid X > 15) = \frac{\mathbb{P}(X>25)}{\mathbb{P}(X>15)} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.

10. A stick of length 1 is broken at a uniformly random point, yielding two pieces. Let X and Y be the lengths of the shorter and longer pieces, respectively, and let $R = X/Y$ be the ratio of the lengths X and Y.

    (a) Find the CDF and PDF of R.

    (b) Find the expected value of R.

Solution:

(a) Let U be the uniform random variable in $[0, 1]$. Then,

$$R = \begin{cases} \frac{U}{1-U}, & 0 \leqslant U \leqslant \frac{1}{2}, \\ \frac{1-U}{U}, & \frac{1}{2} < U \leqslant 1. \end{cases}$$

Thus, $F(x) = 0$ for $x \leqslant 0$,

$$F(x) = \mathbb{P}(R \leqslant x) = \mathbb{P}\left(\frac{U}{1-U} \leqslant x, U \leqslant \frac{1}{2}\right) + \mathbb{P}\left(\frac{1-U}{U} \leqslant x, U > \frac{1}{2}\right) = \frac{2x}{1+x}$$

for $0 \leqslant x \leqslant 1$, and $F(x) = 1$ for $x \geqslant 1$. Taking the derivative in x, the pdf is $f(x) = 2(1+x)^{-2}$ for $0 \leqslant x \leqslant 1$ and otherwise $f(x) = 0$.

(b)

$$\mathbb{E}[R] = \int_0^1 \frac{2x}{(1+x)^2}\, dx = 1/6.$$