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SLLN : $\frac{S_n}{n} \to \mu$ w/ prob. 1 , That is

$$\mathbb{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mu \right) = 1$$

Proof Suppose $\mu = 0$ (Redefine X_i by $X_i - \mu$) and

$\mathbb{E}[X_i^4] < \infty$ (can be removed by " truncation argument ")

Borel – Cantelli Lemma : $\mathbb{P}\left(\lim_{n\to\infty} Y_n = Y \right) = 1$

$\Leftrightarrow$ $\mathbb{P}\left(\lim_{n\to\infty} Y_n \neq Y \right) = 0$

$\Leftrightarrow$ For $\varepsilon > 0$, $\mathbb{P}(|Y_n - Y| > \varepsilon$ infinitely many $n) = 0$

$\Leftrightarrow$ For $\varepsilon > 0$, $\mathbb{P}(\forall m , \exists\, n \geq m$ s.t. $|Y_n - Y| > \varepsilon) = 0$.

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\mathbb{P}\left(\bigcap_{m=1}^{\infty} \bigcup_{n\geq m} \{ |Y_n - Y| > \varepsilon \} \right) = 0$

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\lim_{m\to\infty} \mathbb{P}\left(\bigcup_{n\geq m} \{ |Y_n - Y| > \varepsilon \} \right) = 0$

If $\sum_{m=1}^{\infty} \mathbb{P}(|Y_m - Y| > \varepsilon) < \infty$ then

$$\mathbb{P}\left(\bigcup_{n\geq m} \{ |Y_n - Y| > \varepsilon \} \right) \leq \sum_{n=m}^{\infty} \mathbb{P}(|Y_n - Y| > \varepsilon) \longrightarrow 0$$

By this, it suffices to show that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) < \infty$$

To this end,

$$\mathbb{P}\left(\left| \frac{S_n}{n} \right| > \varepsilon \right) \leq \frac{\mathbb{E}\, S_n^4}{n^4 \varepsilon^4} \leq \frac{C}{n^2 \varepsilon^4} .$$

$\blacksquare$

Def $X_n \to X$ in prob. if $\mathbb{P}(|X_n - X| > \varepsilon) \to 0 \quad \forall \varepsilon$.

$X_n \to X$ a.s. if $\mathbb{P}(\lim_{n\to\infty} X_n = X) = 1$

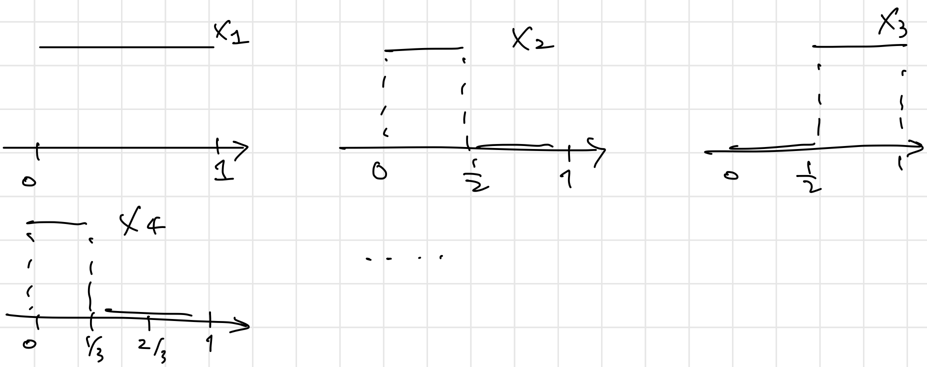
Prp $X_n \to X$ a.s implies $X_n \to X$ in prob.

Proof

$$\lim_{m\to\infty} \underline{\mathbb{P}\left(\bigcup_{n \geq m} \{|X_n - X| > \varepsilon\}\right)} = 0$$

$$\geq \mathbb{P}(|X_m - X| > \varepsilon).$$

Note The converse is not true.



$X_n \to 0$ in prob.

$X_n \not\to 0$ a.s.

$\boxed{\text{Central Limit Thm}}$

Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables with $\mu = \mathbb{E}X_1$, $\sigma^2 = Var(X_1) < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$, then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - E[S_n]}{\sqrt{Var(S_n)}} \leq x\right) = \mathbb{P}(Z \leq x) = \Phi(x), \quad \forall x \in \mathbb{R}$$

Let $\mu = EX_i$, $\sigma^2 = Var(X_i)$ then $ES_n = \mu n$, $Var(S_n) = n\sigma^2$

Thus

$$\mathbb{P}\left(\frac{S_n - E[S_n]}{\sqrt{Var(S_n)}} \leq x\right) = \mathbb{P}\left(\frac{S_n - \mu \cdot n}{\sigma\sqrt{n}} \leq x\right)$$