Final Take-Home Exam

MATH 461 Spring 2022
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Due date: May 12, 2022 at 8pm

1. Consider N distinct points on a circle, say $1, 2, \cdots, N$. For each distinct points i, j on the circle, we connect the points with a line segment with probability $p \in [0, 1]$ independently. In other words, we consider a collection of independent Bernoulli random variables $\{X_{\{i,j\}}\}_{\{i,j\}\subset\{1,\cdots,N\}}$ with probability p. If $X_{\{i,j\}} = 1$ then we connect the points i and j. (See the pictures below.)
   (a) (5 points) Let S be the number of edges. Find $\mathbb{E}[S]$ and $\mathrm{Var}(S)$.
   (b) (5 points) Let T be the number of triangles. Find $\mathbb{E}[T]$.
   (c) (Extra credit: 4 points) Compute $\mathrm{Var}(T)$.
2. Let A, B be independent exponential random variables with parameters 1 and 2 respectively.
   (a) (5 points) Consider an equation $x^2 - Ax + B = 0$. Find the probability that the equation has two distinct real roots. (Hint: A quadratic equation $ax^2 + bx + c = 0$ has two distict real roots if and only if $b^2 - 4ac > 0$.)
   (b) (5 points) Consider $x^2 - Ax - B = 0$. In this case, the equation has real roots with probability 1. Since $-B < 0$, one root is positive and the other is negative. Let P be the positive root and Q be the negative root. Find the joint density function for P and Q. (Hint: Using $x^2 - Ax - B = (x - P)(x - Q)$, one can describe A, B in terms of P, Q.)
   (c) (Extra credit: 4 points) Find the marginal density for P.
3. (10 points) The number of students who enroll in a psychology course is a Poisson random variable with mean 100. The professor in charge of the course has decided that if the number enrolling is 120 or more, he will teach the course in two separate sections, whereas if fewer than 120 students enroll, he will teach all of the students together in a single section. Find the approximated probability that the professor will have to teach two sections using the Central Limit Theorem.

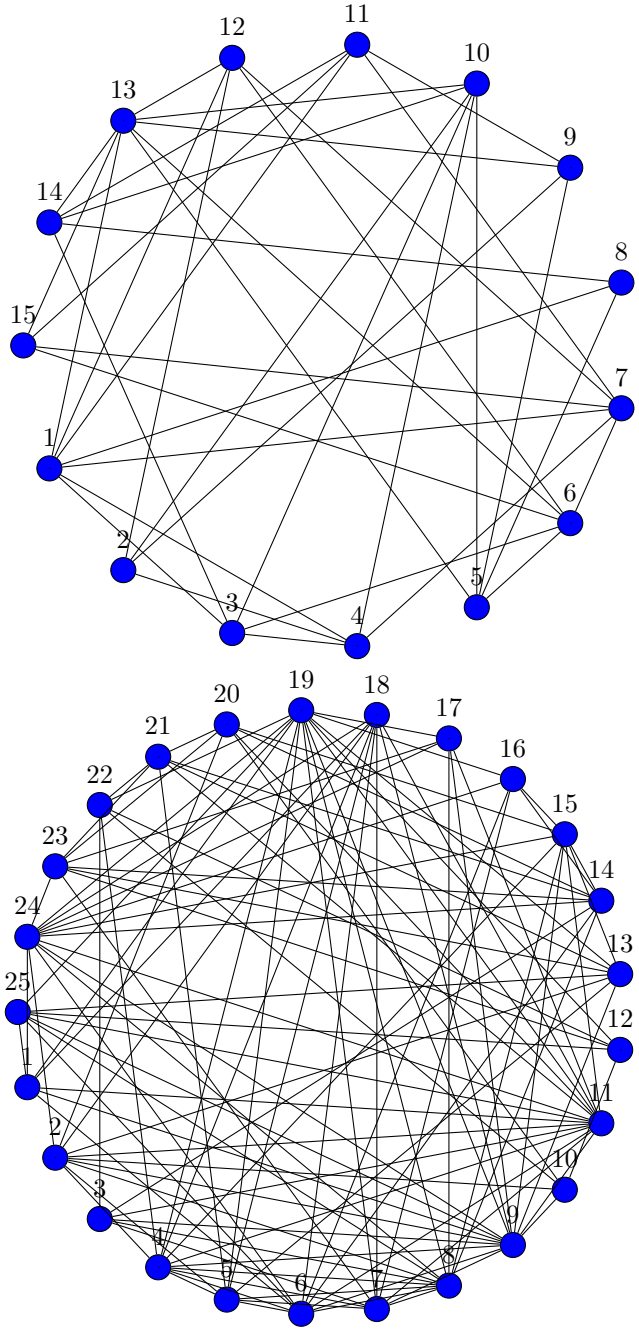

Figure 1: Examples of Problem 1 for $N = 15$ with $p = 0.4$ and $N = 25$ with $p = 0.3$.