MIDTERM EXAM 3

DATE : Nov. 8 (WED)

TIME : 6:30 PM — 7:45PM

PLACE : SECTION A — BOGGS B5
(8:25)

SECTION E — HOWEY-PHYSICS
(11:00) L3

COVERAGE: UP TO 6.6.

REVIEW : Nov 8 in CLASS
(SAMPLE EXAMS : S22, F22,
S23)
↑
MASTER WEBPAGE.

Math 1554 Linear Algebra Spring 2023

Midterm 3

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ________________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Kim Prof Barone Prof Schroeder Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (6 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If the projection of vector $\vec{y}$ onto subspace W is equal to the zero vector, then $\vec{y} \in W^\perp$.
○	○	If A and B are $n \times n$ orthogonal matrices, then AB is also $n \times n$ and orthogonal.
○	○	If $\{\vec{u}, \vec{v}\}$ is an orthonormal set in $\mathbb{R}^n$, then $\|\vec{u} + \vec{v}\| = \sqrt{2}$.
○	○	If A is row equivalent to a diagonalizable matrix B, then A is diagonalizable.
○	○	For any rectangular $m \times n$ matrix A, $(\text{Row}A)^\perp = (\text{Row}A^TA)^\perp$.
○	○	If A has the QR factorization $A = QR$, then $\text{Col}A = \text{Col}Q$.

(b) (2 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A 2×2 real matrix A with eigenvalues $1 + i$ and $-1 - i$.
○	○	A diagonalizable matrix A that is similar to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

(c) (2 points) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for subspace V. Fill in circles next to orthogonal bases for V; leave the other circles empty.

○ $\{\vec{v}_1, \vec{v}_2\}$

○ $\{\vec{v}_3, 4\vec{v}_1, 3\vec{v}_2\}$

○ $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2, \vec{v}_3\}$

○ $\{\vec{v}_1, \vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_2 + \vec{v}_3\}$

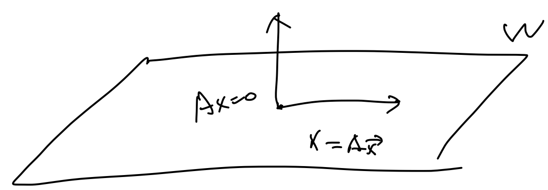
(d) (2 points) Let W be a 4-dimensional subspace of $\mathbb{R}^5$ and let A be the standard matrix for the orthogonal projection onto W. The following situations are either possible or impossible. Fill in the circles next to the **possible** situations; leave the other circles empty.

○ A is invertible.

○ A has eigenvalue zero.

○ Nul$A = W$. $\perp$

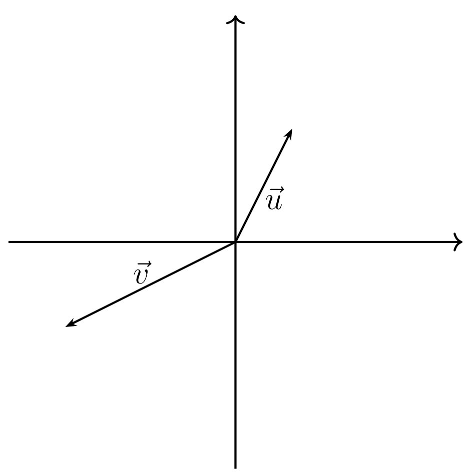
○ $Av = v$, for some vector $v \in \mathbb{R}^5$.



W

$$A = P \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} P^T$$

2. (1 point) In the graph below, sketch $\text{proj}_{\vec{u}}(\vec{v})$.



Math 1554 Linear Algebra, Midterm 3. Your initials: ________________

You do not need to justify your reasoning for questions on this page.

3. (9 points) Fill in the blanks.

   (a) If A is 20×25 and dim(Col$(A)^{\perp}$) = 10, the rank of A is $\boxed{}$.

   (b) If A is an $n \times n$ orthogonal matrix, $A^T A$ is equal to $\boxed{}$.

   (c) The distance between vector $\vec{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and subspace $W = \text{Span}(\vec{v})$, where $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is equal to $\boxed{}$.

   (d) If u and v are orthogonal vectors in $\mathbb{R}^n$ and the columns of $A \in \mathbb{R}^{n \times n}$ are orthonormal, then $(Au) \cdot (Av)$ is equal to $\boxed{}$ (a number).

   (e) If $\vec{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $W = \text{Span}(\vec{u})$, then $\text{proj}_{\vec{u}}\vec{b}$ is the vector $\begin{pmatrix} \\ \end{pmatrix}$.

   (f) If $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $W = \text{Span}(\vec{v})$, a basis for $W^{\perp}$ is the vector $\begin{pmatrix} \\ \end{pmatrix}$.

   (g) If A is 32×9 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{32}$, then the number of pivot columns in A is $\boxed{}$.

   (h) If A is 30×8 and $\dim(\text{Row}(A)^{\perp})$ = 1, then rank A is equal to $\boxed{}$.

   (i) If $W = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$, then $\dim W^{\perp} = \boxed{}$.

Math 1554 Linear Algebra, Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

4. (4 points) If possible, give examples of the following. If it is not possible, write `NP`.

   (a) A 4×5 non-zero matrix, A, in RREF, that satisfies $\dim\left(\underbrace{(\text{Row}(A))^{\perp}}_{= \text{Null}(A)}\right) = 3$.

   $$A = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

   3×2

   $\begin{bmatrix} 1 & -2 \\ \vdots & \\ \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$

   (b) A 2×3 matrix such that $\underbrace{\text{Col}(A)^{\perp}}_{= \text{Null}(A^T)}$ is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   $$A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -4 & -6 \end{pmatrix}$$

   (c) A non-zero vector, $\vec{w}$, whose projection onto the space spanned by $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is the zero vector.

   $$\vec{w} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

   (d) A 3×2 matrix A that is in echelon form, has QR factorization $A = QR$, and $A = Q$.

   $$A = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

5. (2 points) W is the subspace spanned by $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$. Give a basis for $W^{\perp}$.

You do not need to justify your reasoning for questions on this page.

6. (2 points) Given that $\vec{u}_1$ and $\vec{u}_2$ form an orthogonal set, compute the projection of $\vec{y}$ onto S = Span$\{\vec{u}_1, \vec{u}_2\}$.

$$\vec{u}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$$

proj$_S(\vec{y})$= $\boxed{}$

7. (3 points) If $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and $\lambda_1 = 2 + i$, find λ_2, v_1, v_2.

λ_2= $\boxed{}$

$\vec{v}_1$= $\boxed{}$

$\vec{v}_2$= $\boxed{}$

8. (4 points) **Show all work for problems on this page.** If $A = QR = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, determine the least-squares solution to $A\hat{x} = \begin{pmatrix} \sqrt{2} \\ 2\sqrt{2} \end{pmatrix}$. You do not need to determine A.

$\hat{x}$= $\boxed{}$

9. (5 points) **Show all work for problems on this page.** Four points in $\mathbb{R}^3$ with coordinates (x, y, z) are $(2, 0, 1)$, $(0, -2, 1)$, $(1, -1, 3)$, and $(1, 1, 3)$. Determine the coefficients c_1 and c_2 for the plane $z = c_1 x + c_2 y$ that best fits the points using the method of least-squares.

$$c_1 = \boxed{} \qquad c_2 = \boxed{}$$

10. (4 points) **Show all work for problems on this page.** Write $\vec{y}$ as the sum of a vector $\hat{y}$ in Span$\{\vec{u}\}$ and a vector $\vec{z}$ orthogonal to $\vec{u}$.

$$\vec{y} = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 1 \end{pmatrix}, \qquad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \hat{y} = \boxed{} \qquad \vec{z} = \boxed{}$$

11. (4 points) Find matrices P and D such that $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$P = \boxed{}$

$D = \boxed{}$

Math 1554 Linear Algebra Fall 2022

Midterm 3

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ____________________

Student GT Email Address: ________________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name ________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler
Prof Blumenthal Prof Sun Prof Shirani

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	A matrix $A \in \mathbb{R}^{n\times n}$ and its transpose A^{T} have the same eigenvectors.
◯	◯	An invertible matrix A is diagonalizable if and only if its inverse A^{-1} is diagonalizable.
◯	◯	If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, then $\vec{u}$ is orthogonal to $(\vec{w} - \vec{v})$.
◯	◯	If the vectors $\vec{u}$ and $\vec{v}$ are orthogonal then $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.
◯	◯	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\| \operatorname{proj}_W(\vec{y})\|$ is the shortest distance between W and $\vec{y}$.
◯	◯	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\vec{y} - \operatorname{proj}_W(\vec{y})$ is in $W^{\perp}$.
◯	◯	If W is a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ such that $\vec{y} \cdot \vec{w} = 0$ for some vector $\vec{w} \in W$, then $\vec{y} \in W^{\perp}$.
◯	◯	The line of best fit $y = \beta_0 + \beta_1 x$ for the points $(1, 2), (1, 3)$, and $(1, 4)$ is unique.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A 5×5 real matrix A such that A has no real eigenvalues.
◯	◯	An $m \times n$ matrix U where $U^T U = I_n$ and $n > m$.
◯	◯	A 2-dimensional subspace W of $\mathbb{R}^3$ and a vector $\vec{y} \in W$ such that $\|\vec{v}_1 - \vec{y}\| = \|\vec{v}_2 - \vec{y}\|$ where $\vec{v}_1, \vec{v}_2 \in W^{\perp}$ and $\vec{v}_1 \neq \vec{v}_2$.
◯	◯	A matrix $A \in \mathbb{R}^{3\times 4}$ such that the linear system $A\vec{x} = \vec{b}$ has a unique least-squares solution.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) An $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ has non-zero orthogonal columns and $A^T A = 2I_n$. Which of the following statements is FALSE?

- ○ $(A\vec{x})\cdot(A\vec{y}) = \vec{x}\cdot\vec{y}$ for every $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.
- ○ $n \leq m$.
- ○ If we apply the Gram-Schmidt process to $\{\vec{a}_1, \ldots, \vec{a}_n\}$ we obtain the same set $\{\vec{a}_1, \ldots, \vec{a}_n\}$.
- ○ If $A = QR$ is the QR factorization of A, then R is a diagonal matrix.

2. (3 points) Find a, b, c so that the set of vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} -2 \\ 0 \\ a \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} -1 \\ b \\ c \end{pmatrix}$$

$a =$ $\boxed{}$

$b =$ $\boxed{}$

$c =$ $\boxed{}$

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

3. (8 points) Fill in the blanks.

   (a) Suppose $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ and that $\vec{v}$ is a unit vector. If $(2\vec{u}+\vec{v})\cdot(\vec{u}+5\vec{v}) = 13$, determine the length of $\vec{u}$. $\|\vec{u}\| =$ $\boxed{}$

   (b) The normal equations for the least-squares solution to $A\vec{x} = \vec{b}$ are given by: $\boxed{}$

   (c) Compute the length (magnitude) of the vector $\vec{y}$.

$$\vec{y} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$\|\vec{y}\| =$ $\boxed{}$

   (d) Let $A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$. The vector $\vec{v} = \begin{pmatrix} -1-i \\ -1+i \end{pmatrix}$ is an eigenvector of A. Find the associated eigenvalue λ for the eigenvector $\vec{v}$ of A. $\lambda =$ $\boxed{}$

Midterm 3. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

4. (4 points) Fill in the blanks.

(a) Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ and let $\vec{y} = \begin{pmatrix}4\\5\\6\end{pmatrix}$. Calculate the projection of $\vec{y}$ onto the subspace W, and find the distance from $\vec{y}$ to W.

$\text{proj}_W(\vec{y})=$ $\boxed{}$

$\text{dist}(\vec{y}, W) =$ $\boxed{}$

(b) Let $W = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$, $L = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Find **all** vectors $\vec{u} \in L$ such that the distance from $\vec{u}$ to W is equal to 2.

$\boxed{}$

5. (6 points) **Show all work for problems on this page.**

   One of the eigenvalues of the matrix A is $\lambda = 1$. Diagonalize the matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$P =$ $\boxed{}$

$D =$ $\boxed{}$

6. (3 points) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 4$, and $\vec{v}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = -3$.

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$A =$ $\boxed{}$

7. (4 points) **Show all work for problems on this page.**
Let $\mathcal{B} = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ be a basis for a subspace W of $\mathbb{R}^4$, where

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}.$$

(a) Apply the Gram-Schmidt process to the set of vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ to find an orthogonal basis $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for W. *Clearly show all steps of the Gram-Schmidt process.*

$\mathcal{H}$= $\boxed{}$

(b) In the space below, **check** that the vectors in the basis $\mathcal{H}$ form an orthogonal set.

8. (4 points) **Show all work for problems on this page.**
If A has the following QR factorization

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 0 & 3 \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix},$$

compute the least-square solution to the equation $A\vec{x} = \vec{b}$.

$\hat{x} =$ $\boxed{}$

9. (4 points) Compute the least squares line $y = c_1 + c_2 x$ that best fits the data

x	-1	0	1
y	4	2	5

$\boxed{}$

Math 1554 Linear Algebra   Spring   2022

Midterm 3

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Barone   Prof Shirani   Prof Simone   Prof Timko

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

$A\vec{x} = \text{proj}_{Col(A)}(b)$

$A\vec{y} = \text{proj}_{Col(A)}(b)$

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.
○	○	If W is the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, then $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a vector in $W^\perp$.
○	○	$U^T \cdot U = I$ If U is a 3×2 matrix with orthonormal columns, then for every $\vec{y} \in \text{Col}(U)$ we have $\vec{y} = UU^T\vec{y}$.
○	○	If the matrix A has orthogonal columns, then A^TA is a diagonal matrix.
○	○	Assume $n \neq m$. If $A = QR$ is the QR factorization of $A \in \mathbb{R}^{n \times m}$, then $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{n \times m}$.
○	○	If $A = QR$ is a QR factorization of A, then $A^TA = R^TR$.
●	○	If $\hat{x}$ and $\hat{y}$ are ~~least squares~~ solutions of $A\vec{x} = \vec{b}$, then $\hat{x} - \hat{y} \in \text{Nul}(A)$.
○	○	Suppose A is such that T_A is not one-to-one, and $\vec{b}$ is not in the range of T. Then $A\vec{x} = \vec{b}$ has a unique least-squares solution.

$A^TA\vec{x} ? A^Tb$

$A^TA\vec{y} = A^Tb$

$A^TA(x-y) = 0$

$x - y \in Nul(A^TA)$

$\shortparallel$

$Nul(A)$

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A is a 7×7 diagonalizable matrix with exactly three distinct eigenvalues whose geometric multiplicities are 1, 2, and 3, respectively.
○	○	$\vec{u}$ and $\vec{v}$ are nonzero vectors such that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.
○	○	The distance between a vector $\vec{b} \in \mathbb{R}^m$ and the column space of a matrix $A \in \mathbb{R}^{m \times n}$ is zero, and the linear system $A\vec{x} = \vec{b}$ is inconsistent.
○	○	$\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$, and there exists a linearly independent set of vectors $\{\vec{x}, \vec{y}\}$ in $\mathbb{R}^3$ such that $\text{Proj}_{\mathcal{W}}\vec{x} = \text{Proj}_{\mathcal{W}}\vec{y}$.

$$U = [\, u_1 \quad u_2 \,] \in \mathbb{R}^{3\times 2}$$

$$U \cdot U^T \cdot \vec{y} = [\, u_1 \quad u_2 \,] \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \cdot y$$

$$\begin{bmatrix} u_1 \cdot y \\ u_2 \cdot y \end{bmatrix}$$

$$= (u_1 \cdot y) \cdot u_1 + (u_2 \cdot y) \cdot u_2 \quad \cancel{=} \; y$$

$y \in Col(U) \Rightarrow \quad y = c_1 \cdot u_1 + c_2 \cdot u_2$

$y = \underline{U \cdot U^T \cdot y}$

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = y \cdot u_1$$

$$c_2 = y \cdot u_2$$

$$U \cdot U^T y = \text{proj}_{Col(U)}(y)$$

Midterm 3. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

(c) (2 points) The standard matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has orthonormal columns. Which one of the following statements is **false**?

Choose only one.

$T(\vec{x}) = Ax$

- ○ $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^3$. T.
- ○ If two non-zero vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$ are scalar multiples of each other, then $\|T(\vec{x} + \vec{y})\|^2 = \|T(\vec{x})\|^2 + \|T(\vec{y})\|^2$. $y = 2\vec{x}$
- ○ If $\mathcal{P}$ is a parallelpiped in $\mathbb{R}^3$, then the volume of $T(\mathcal{P})$ is equal to the volume of $\mathcal{P}$.
- ○ T is one-to-one.

$\mathrm{Vol}(T(P)) = \underbrace{|\det(A)|}_{= 1?} \cdot \mathrm{Vol}(P)$

$A^T A = I$

$\det(A^T \cdot A) = 1$

$= \det(A^T) \cdot \det(A)$

$= (\det(A))^2$

$\|Ax\| = \|x\|$

$\|3T(\vec{x})\|^2 = \|T(x)\|^2 + \|2 \cdot T(x)\|^2$

$9\|T(x)\|^2 = \|T(\vec{x})\|^2 + 4\|T(\vec{x})\|^2$

2. (2 points) Suppose that, in the QR factorization of A, we have Q as given below. Find R.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad Q = \frac{1}{2}\begin{bmatrix} 1 & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \\ 1 & -\sqrt{3} \\ 1 & 1/\sqrt{3} \end{bmatrix}$$

Note: Please fill in the blanks and do not place values in front of the matrix for this problem.

$$R = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

You do not need to justify your reasoning for questions on this page.

3. (2 points) Using **only** 0's and 1's in your answer, give an example of a 2×2 matrix that is invertible but not diagonalizable.

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

4. (6 points) Fill in the blanks.

   (a) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be orthogonal vectors each with length 2. Determine the length of the vector $2\vec{u} + \vec{v}$. $\boxed{}$

   (b) Suppose A is a 7×5 matrix such that dim $(\text{Row } A)^{\perp} = 4$. Determine the dimension of the column space of A. $\boxed{}$

   (c) Suppose $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$, and $\vec{x}$ belongs to the subspace $\mathcal{W}$. Suppose also that

$$\vec{v}_1 \cdot \vec{v}_1 = 2, \vec{v}_2 \cdot \vec{v}_2 = 4, \vec{v}_1 \cdot \vec{x} = 6, \text{ and } \vec{v}_2 \cdot \vec{x} = -4.$$

   Find $[\vec{x}]_{\mathcal{B}}$ the coordinates of $\vec{x}$ in the basis $\mathcal{B}$. $\boxed{}$

Midterm 3. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

5. (6 points) Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be an orthogonal basis for $\mathbb{R}^3$, where

$$\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^3$ that is spanned by $\vec{u}_1$ and $\vec{u}_2$.

(a) Give an orthonormal basis for $\mathcal{W}^\perp$.

(b) What is the closest vector in $\mathcal{W}$ to the vector $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$?

(c) Give a vector $\vec{x} \in \mathbb{R}^3$ that satisfies all of the following conditions:

- $\vec{x}$ is orthogonal to $\vec{u}_3$.
- $\text{Proj}_{\mathcal{L}_1}\vec{x} = 2\vec{u}_1$, where $\mathcal{L}_1 = \text{span}\{\vec{u}_1\}$.
- $\text{Proj}_{\mathcal{L}_2}\vec{x} = 3\vec{u}_2$, where $\mathcal{L}_2 = \text{span}\{\vec{u}_2\}$.

6. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

(i) List the eigenvalues of A. $\boxed{}$

(ii) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If this is not possible, write `NP`.

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

7. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

   Find an orthogonal basis of the subspace spanned by the set of vectors shown below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

   Note: In order to receive full credit you must clearly show all steps of the Gram-Schmidt process applied to the vectors.

8. (8 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

In this problem, you will use the least-squares method to find the values α and β which best fit the curve

$$y = \alpha \cdot \frac{1}{1+x^2} + \beta$$

to the data points $(-1,1), (0,-1), (1,0)$ using the parameters α and β.

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?

(ii) What is the augmented matrix for the normal equations for this system.

(iii) Find a least-squares solution to the linear system from (i) to determine the parameters α and β of the best fitting curve.

$\alpha =$ $\qquad$ $\beta =$

Math 1554 Linear Algebra Spring 2023

Midterm 3 Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ________________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Kim Prof Barone Prof Schroeder Prof Kumar

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

You do not need to justify your reasoning for questions on this page.

1. (a) (6 points) Suppose A is a real $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	For any line $L \in \mathbb{R}^2$ passing through the origin, the matrix corresponding to the transformation that reflects across the line L must always be diagonalizable.
○	○	If A and B are $n \times n$ orthogonal matrices, then AB is also $n \times n$ and orthogonal.
○	○	If A is the reduced row echelon form (RREF) of B and A is diagonalizable, then B is diagonalizable.
○	○	If $\vec{b} \in \text{Col}(A)$, then the least squares solution to the linear system $A\vec{x} = \vec{b}$ is unique.
○	○	For any rectangular $m \times n$ matrix A, $(\text{Nul}A)^{\perp} = \text{Row}A^TA$.
○	○	If the distance of $\vec{w}$ from $\vec{v}$ is equal to the distance of $\vec{w}$ from $-\vec{v}$, then $\vec{w} \cdot \vec{v} = 0$.

(b) (2 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A diagonal matrix A that is similar to $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.
○	○	An orthogonal matrix A such that $\lvert \det A \rvert \neq 1$.

Midterm 3 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

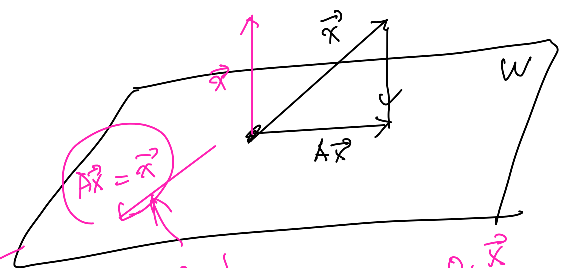
(c) (2 points) $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis for subspace V. Fill in circles next to orthogonal bases for V; leave the other circles empty.

- ◯ $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 + \vec{v}_3, \vec{v}_2 + \vec{v}_3\}$
- ◯ $\{\vec{v}_3, 2\vec{v}_1, -\vec{v}_2\}$
- ◯ $\{\vec{v}_1 + \vec{v}_2, \vec{v}_1 - \vec{v}_2, \vec{v}_3\}$
- ◯ $\{\vec{v}_1, \vec{v}_2 - \vec{v}_3\}$

$A\vec{x} = \lambda\vec{x}$

(d) (2 points) Let W be a 3-dimensional subspace of $\mathbb{R}^6$ and let A be the standard matrix for the orthogonal projection onto W. The following situations are either possible or impossible. Fill in the circles next to the **possible** situations; leave the other circles empty.

- ◯ None of the eigenvalues of A are zero.
- ◯ A is singular.
- ◯ $A\vec{v} = \vec{0}$, for some non-zero vector $v \in W$.
- ◯ $\dim \text{Nul}A = \dim \text{Col}A$. $= 3$



$Av = v \neq 0$

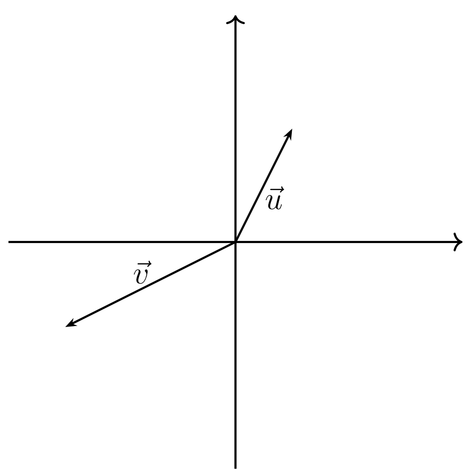
$\lambda = 1$

$A\vec{x} = 0 \cdot \vec{x}$

$\text{Nul}(A) = W^{\perp}$

$\text{Col}(A) = W$

2. (1 point) In the graph below, sketch $\text{proj}_{\vec{v}}(\vec{u})$.



3. (9 points) Fill in the blanks.

(a) If A is 15×17 and dim(Col$(A)^{\perp}$) = 10, the rank of A is $\boxed{}$.

(b) The distance between vector $\vec{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and subspace $W = \text{Span}(\vec{v})$, where $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is equal to $\boxed{}$.

(c) If $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $v_1 v_2^T$ is the matrix $\begin{pmatrix} & \\ & \end{pmatrix}$.

(d) The distance between the vector $\vec{u} = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -4 \\ -1 \\ 8 \end{pmatrix}$ is equal to $\boxed{}$.

(e) If $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$, then $\dim W^{\perp} = \boxed{}$.

4. (4 points) If possible, give examples of the following. If it is not possible, write `NP`.

   (a) A 4×5 matrix A with Col$A = $ span$\{[1\ 1\ 1\ 1]^T\}$, that satisfies $\dim\left((\text{Row}A)^\perp\right) = 4$.

   $$A = \begin{pmatrix} & & & & \\ & & & & \\ & & & & \end{pmatrix}$$

   (b) A 2×3 matrix such that Col$(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

   $$A = \begin{pmatrix} & & \\ & & \end{pmatrix}$$

   (c) A non-zero vector, $\vec{w}$, whose projection onto the $W = $ span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$ is the zero vector.

   $$\vec{w} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

   (d) A 3×2 matrix A that is in echelon form, has QR factorization $A = QR$, and $A = Q$.

   $$A = \begin{pmatrix} & \\ & \\ & \end{pmatrix}$$

5. (2 points) W is the subspace spanned by $\vec{u} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Give a basis for $W^\perp$.

6. (5 points) **Show all work for problems on this page.** Let $W = \mathrm{Nul}(A)$ where

$$A = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \vec{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}.$$

(a) Find $\mathrm{proj}_{W^\perp}(\vec{y})$. *Please box your answer.*

(b) Find $\vec{y} = \vec{u} + \vec{v}$ where $\vec{u} \in W$ and $\vec{v} \in W^\perp$. *Please box your answer.*

7. (4 points) **Show all work for problems on this page.** Compute the equation of the line $y = \beta_0 + \beta_1 x$ of best fit using the least-squares method for the data below.
Please box your answer.

x	-1	0	1
y	0	1	5

8. (5 points) **Show all work for problems on this page.** Let $\mathcal{B} = \{x_1, x_2, x_3\}$ be a basis for a subspace W of $\mathbb{R}^4$ where

$$\vec{x}_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 6 \\ -8 \\ -2 \\ -4 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 6 \\ 3 \\ 6 \\ -3 \end{pmatrix}.$$

Apply the Gram–Schmidt process to the set $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ to find an orthonormal basis $\mathcal{H} = \{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$ for the subspace W. *Please box your answer.*

9. (4 points) **Show all work for problems on this page.** If $A = \begin{pmatrix} -5 & -5 \\ 5 & -5 \end{pmatrix}$ has eigenvalues λ_1, λ_2 with corresponding eigenvectors v_1, v_2, and $\lambda_1 = -5 + 5i$, find λ_2, v_1, v_2.

λ_2= $\boxed{}$

$\vec{v}_1$= $\boxed{}$

$\vec{v}_2$= $\boxed{}$

10. (4 points) Find matrices P and D such that $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$$

P= $\boxed{}$

D= $\boxed{}$

Math 1554 Linear Algebra Fall 2022

Midterm 3 Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler

Prof Blumenthal Prof Sun Prof Shirani

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

Midterm 3 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	A matrix $A \in \mathbb{R}^{n\times n}$ and its transpose A^{T} have the same eigenvectors.
◯	◯	The line of best fit $y = \beta_0 + \beta_1 x$ for the points $(1,1), (2,1),$ and $(3,1)$ is unique.
◯	◯	If the vectors $\vec{u}$ and $\vec{v}$ are orthogonal then $\\|\vec{u}+\vec{v}\\| = \\|\vec{u}\\| + \\|\vec{v}\\|$.
◯	◯	A triangular matrix A is diagonalizable if and only if A is invertible.
◯	◯	If $A = PDP^{-1}$ where D is a diagonal matrix, then D and A have the same eigenvectors.
◯	◯	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\mathrm{proj}_W(\vec{y})$ is in W.
◯	◯	If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and W is a subspace of $\mathbb{R}^n$, then $\\|\vec{y} - \mathrm{proj}_W(\vec{y})\\|$ is the shortest distance between W and $\vec{y}$.
◯	◯	If W is a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ such that $\vec{y}\cdot\vec{w} = 0$ for some vector $\vec{w} \in W$, then $\vec{y} \in W^{\perp}$.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A 3×3 real matrix A such that A has eigenvalues $2, 3, 2i + 3$.
◯	◯	An $m \times n$ matrix U where $U^T U = I_n$ and $n > m$.
◯	◯	A 2-dimensional subspace W of $\mathbb{R}^3$ and a vector $\vec{y} \in W$ such that $\\|\vec{v}_1 - \vec{y}\\| = \\|\vec{v}_2 - \vec{y}\\|$ where $\vec{v}_1, \vec{v}_2 \in W^{\perp}$ and $\vec{v}_1 \neq \vec{v}_2$.
◯	◯	A vector $\vec{y} \in \mathbb{R}^3$ and a subspace W in $\mathbb{R}^3$ such that $\vec{y} = \vec{w} + \vec{z}$ where $\vec{w}$ is in W, but $\vec{z}$ is not in $W^{\perp}$.

Midterm 3 Make-up. Your initials: ______________

$a_1, \dots, a_n \in \mathbb{R}^m$ lin. indep. $\Rightarrow$ $n \le m$

You do not need to justify your reasoning for questions on this page.

(c) (2 points) An $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ ($\in \mathbb{R}^m$) has non-zero orthogonal columns and $A^T A = 2I_n$. Which of the following statements is FALSE?

- ⊗ $(A\vec{x})\cdot(A\vec{y}) = \vec{x}\cdot\vec{y}$ for every $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.
- ○ $n \le m$. ✓
- ○ If we apply the Gram-Schmidt process to $\{\vec{a}_1, \dots, \vec{a}_n\}$ we obtain the same set $\{\vec{a}_1, \dots, \vec{a}_n\}$ ✓
- ○ If $A = QR$ is the QR factorization of A, then R is a diagonal matrix.

$$(A\vec{x})\cdot(A\vec{y}) = (A\vec{x})^T\cdot(Ay) = \vec{x}^T\cdot A^T\cdot A\cdot y = 2\cdot x^T\cdot y = 2\,\vec{x}\cdot\vec{y}$$

$$R = \begin{pmatrix} \underset{\|a_1\|}{a_1\cdot u_1} & \overset{=0}{a_2\cdot u_1} & \cdots \\ 0 & \underset{\|a_2\|}{a_2\cdot u_2} & \\ 0 & 0 & \\ \vdots & \vdots & \\ 0 & 0 & \end{pmatrix}$$

$$a_1\cdot u_1 = \frac{a_1}{\|a_1\|}\cdot a_1 = \frac{\|a_1\|^2}{\|a_1\|} = \|a_1\|.$$

2. (3 points) Using **only** 0's and 1's in your answers, give an example of 2×2 matrices that satisfy: A is invertible but not diagonalizable, and B is diagonalizable but not invertible.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix} \qquad B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Midterm 3 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

3. (8 points) Fill in the blanks.

(a) Suppose $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^n$ and that $\vec{v}$ is a unit vector. If $(3\vec{u}+\vec{v})\cdot(\vec{u}+4\vec{v})=31$, determine the length of $\vec{u}$.

$\|\vec{u}\| =$ $\boxed{}$

(b) The normal equations for the least-squares solution to $A\vec{x}=\vec{b}$ are given by:

$\boxed{}$

(c) Compute the length (magnitude) of the vector $\vec{y}$.

$$\vec{y} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$\|\vec{y}\| =$ $\boxed{}$

(d) Let $A = \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix}$. The vector $\vec{v} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ is an eigenvector of A. Find the associated eigenvalue λ for the eigenvector $\vec{v}$ of A.

$\lambda =$ $\boxed{}$

You do not need to justify your reasoning for questions on this page.

4. (4 points) Fill in the blanks.

   (a) Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ and let $\vec{y} = \begin{pmatrix}1\\2\\3\end{pmatrix}$. Calculate the projection of $\vec{y}$ onto the subspace W, and find the distance from $\vec{y}$ to W.

   $\text{proj}_W(\vec{y})=$ $\boxed{}$

   $\text{dist}(\vec{y}, W) =$ $\boxed{}$

   (b) Let $W = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$, $L = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. Find **all** vectors $\vec{u} \in L$ such that the distance from $\vec{u}$ to W is equal to 2.

   $\boxed{}$

5. (6 points) **Show all work for problems on this page.**

   One of the eigenvalues of the matrix A is $\lambda = 1$. Diagonalize the matrix.

$$A = \begin{pmatrix} -2 & -3 & -2 \\ 2 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$P =$

$D =$

6. (3 points) Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that $\vec{v}_1$ is an eigenvector of A with eigenvalue $\lambda_1 = 2$, and $\vec{v}_2$ is an eigenvector of A with eigenvalue $\lambda_2 = 3$.

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$A =$

7. (4 points) **Show all work for problems on this page.**
Let $\mathcal{B} = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ be a basis for a subspace W of $\mathbb{R}^4$, where

$$\vec{x}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \qquad \vec{x}_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

(a) Apply the Gram-Schmidt process to the set of vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ to find an orthogonal basis $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ for W. *Clearly show all steps of the Gram-Schmidt process.*

$\mathcal{H}=$ $\boxed{}$

(b) In the space below, **check** that the vectors in the basis $\mathcal{H}$ form an orthogonal set.

8. (4 points) **Show all work for problems on this page.**
If A has the following QR factorization

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix},$$

compute the least-square solution to the equation $A\vec{x} = \vec{b}$.

$\hat{x} =$ $\boxed{}$

9. (4 points) Compute the least squares line $y = c_1 x + c_2$ that best fits the data

x	-1	0	1
y	1	-2	2

$\boxed{}$

Math 1554 Linear Algebra Spring 2022

Midterm 3 Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ______________________ GTID Number: __________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ______________________

Circle your instructor:

Prof Barone Prof Shirani Prof Simone Prof Timko

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	The matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.
◯	◯	If W is the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is a vector in $W^{\perp}$.
◯	◯	If U is a 3×2 matrix with orthonormal columns, then $(UU^T)^2 = UU^T$.
◯	◯	If $A^T A = 3I$, then A has orthogonal columns.
◯	◯	Assume $n \neq m$. If $A = QR$ is the QR factorization of $A \in \mathbb{R}^{m \times n}$, then $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$.
◯	◯	If $A = QR$ is a QR factorization of A, then $A^T A = R^T R$.
◯	◯	If $\hat{x}$ and $\hat{y}$ are least-squares solutions of $A\vec{x} = \vec{b}$, then $\hat{x} - \hat{y} \in \mathrm{Nul}(A)$.
◯	◯	Suppose A is such that for every $\vec{b}$ the system $A\vec{x} = \vec{b}$ has a unique least-squares solution. Then $\det(A^T A) \neq 0$.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A is a real 5×5 diagonalizable matrix with exactly three distinct real eigenvalues whose geometric multiplicities are 1, 1, and 2, respectively.
◯	◯	$\vec{u}$ and $\vec{v}$ are nonzero orthogonal vectors such that $\|\|\vec{u}+\vec{v}\|\|^2 < \|\|\vec{u}\|\|^2 + \|\|\vec{v}\|\|^2$.
◯	◯	The distance between a vector $\vec{b} \in \mathbb{R}^m$ and the column space of a matrix $A \in \mathbb{R}^{m \times n}$ is nonzero, and the linear system $A\vec{x} = \vec{b}$ has a unique least squares solution.
◯	◯	$\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$, and there exists a linearly independent set of vectors $\{\vec{x}, \vec{y}\}$ in $\mathbb{R}^3$ such that $\mathrm{Proj}_{\mathcal{W}}\vec{x} = 0$ and $\mathrm{Proj}_{\mathcal{W}}\vec{y} = 0$.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) The standard matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has orthonormal columns. Which one of the following statements is **false**?

Choose only one.

- ○ If $T(\vec{x}) = A\vec{x}$ then $\det(A) = 1$.
- ○ $T(\vec{x}) \cdot T(\vec{y}) = T(\vec{x} \cdot \vec{y})$ for all $\vec{x}, \vec{y}$ in $\mathbb{R}^3$
- ○ $\|T(\vec{x})\| = \|\vec{x}\|$ for all $\vec{x}$ in $\mathbb{R}^3$.
- ○ T is onto.

2. (2 points) Suppose that, in the QR factorization of A, we have Q as given below. Find R.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 2 \\ 1 & 2 \\ -1 & -2 \end{bmatrix} \qquad Q = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ 1 & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \\ -1 & -1/\sqrt{3} \end{bmatrix}$$

Note: Please fill in the blanks and do not place values in front of the matrix for this problem.

$$R = \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

You do not need to justify your reasoning for questions on this page.

3. (2 points) Using **only** 0's and 1's in your answer, give an example of a 2×2 matrix that is not invertible and not diagonalizable. If not possible, write `NP`.

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

4. (6 points) Fill in the blanks.

   (a) Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be vectors each with length 2 and suppose $u \cdot v = -1$. Determine the length of the vector $3\vec{u} + \vec{v}$. $\boxed{}$

   (b) Suppose A is a 5×7 matrix such that dim $(\text{Nul } A)^{\perp} = 4$. Determine the dimension of the orthogonal complement of the column space of A. $\boxed{}$

   (c) Suppose $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for a subspace $\mathcal{W}$ of $\mathbb{R}^n$, and $\vec{x}$ belongs to the subspace $\mathcal{W}$. Suppose also that

$$\vec{v}_1 \cdot \vec{x} = 4, \vec{v}_2 \cdot \vec{x} = -3, \vec{v}_1 \cdot \vec{v}_1 = 2, \text{ and } \vec{v}_2 \cdot \vec{v}_2 = 6.$$

   Find $[\vec{x}]_{\mathcal{B}}$ the coordinates of $\vec{x}$ in the basis $\mathcal{B}$. $\boxed{}$

You do not need to justify your reasoning for questions on this page.

5. (6 points) Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ be an orthogonal basis for $\mathbb{R}^3$, where

$$\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Let $\mathcal{W}$ be a subspace of $\mathbb{R}^3$ that is spanned by $\vec{u}_1$ and $\vec{u}_2$.

(a) Give an orthonormal basis for $\mathcal{W}^\perp$.

(b) What is the closest vector in $\mathcal{W}$ to the vector $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$?

(c) Give a vector $\vec{x} \in \mathbb{R}^3$ that satisfies all of the following conditions:

- $\vec{x}$ is orthogonal to $\vec{u}_3$.
- $\text{Proj}_{\mathcal{L}_1}\vec{x} = 2\vec{u}_1$, where $\mathcal{L}_1 = \text{span}\{\vec{u}_1\}$.
- $\text{Proj}_{\mathcal{L}_2}\vec{x} = 3\vec{u}_2$, where $\mathcal{L}_2 = \text{span}\{\vec{u}_2\}$.

Midterm 3 Make-up. Your initials: ________________

6. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

(i) List the eigenvalues of A. $\boxed{}$

(ii) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If this is not possible, write `NP`.

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \qquad D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

7. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

   Find an orthogonal basis of the subspace spanned by the set of vectors shown below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

   Note: In order to receive full credit you must clearly show all steps of the Gram-Schmidt process applied to the vectors.

8. (8 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

In this problem, you will use the least-squares method to find the values α and β which best fit the curve

$$y = \alpha \cdot 2^{-x^2} + \beta x$$

to the data points $(-1, 2), (0, 0), (1, 1)$ using the parameters α and β.

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?

(ii) What is the augmented matrix for the normal equations for this system.

(iii) Find a least-squares solution to the linear system from (i) to determine the parameters α and β of the best fitting curve.

$\alpha =$ $\qquad$ $\beta =$