Homework 8 Solution

Math 461: Probability Theory, Spring 2022
Daesung Kim

Due date: Apr 1, 2022

1. (a) Let X be the Gamma random variable with $\lambda > 0$ and $\alpha > 0$. Show that $\text{Var}(X) = \frac{\alpha}{\lambda^2}$.
   (b) Let Y be the exponential random variable with parameter $\lambda > 0$. Show that

$$\mathbb{E}\, Y^k = \frac{k!}{\lambda^k} \qquad k = 1, 2, \cdots .$$

Solution:

(a) It follows from $\mathbb{E}[X^2] = \Gamma(\alpha+2)/(\lambda^2\Gamma(\alpha))$ and $\mathbb{E}[X] = \Gamma(\alpha+1)/(\lambda\Gamma(\alpha))$

(b) Since the density of Y is $\lambda e^{-\lambda x}$ on $(0, \infty)$ and otherwise 0, it follows from the change of variables $y = \lambda x$,

$$\mathbb{E}[Y^k] = \int_0^\infty x^k \lambda e^{-\lambda x}\, dx = \lambda^{-k} \int_0^\infty y^k e^{-y}\, dy = \frac{\Gamma(k+1)}{\lambda^k} = \frac{k!}{\lambda^k}.$$

2. Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i-th ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
(a) X_1, X_2;
(b) X_1, X_2, X_3.

Solution: (a)

$\mathbb{P}(X_1 = i, X_2 = j)$	$j = 0$	$j = 1$	$\mathbb{P}(X_1 = i)$
$i = 0$	$\frac{8}{13}\frac{7}{12} = \frac{14}{39}$	$\frac{8}{13}\frac{5}{12} = \frac{10}{39}$	$\frac{24}{39}$
$i = 1$	$\frac{5}{13}\frac{8}{12} = \frac{10}{39}$	$\frac{5}{13}\frac{4}{12} = \frac{5}{39}$	$\frac{15}{39}$
$\mathbb{P}(X_2 = j)$	$\frac{24}{39}$	$\frac{15}{39}$	1

(b)

$$\begin{aligned}
\mathbb{P}(X_1=0,X_2=0,X_3=0) &= \frac{8}{13}\frac{7}{12}\frac{6}{11} = \frac{28}{143}\\
\mathbb{P}(X_1=0,X_2=0,X_3=1) &= \frac{8}{13}\frac{7}{12}\frac{5}{11} = \frac{70}{429}\\
\mathbb{P}(X_1=0,X_2=1,X_3=0) &= \frac{8}{13}\frac{5}{12}\frac{7}{11} = \frac{70}{429}\\
\mathbb{P}(X_1=1,X_2=0,X_3=0) &= \frac{5}{13}\frac{8}{12}\frac{7}{11} = \frac{70}{429}\\
\mathbb{P}(X_1=0,X_2=1,X_3=1) &= \frac{8}{13}\frac{5}{12}\frac{4}{11} = \frac{40}{429}\\
\mathbb{P}(X_1=1,X_2=0,X_3=1) &= \frac{5}{13}\frac{8}{12}\frac{4}{11} = \frac{40}{429}\\
\mathbb{P}(X_1=1,X_2=1,X_3=0) &= \frac{5}{13}\frac{4}{12}\frac{8}{11} = \frac{40}{429}\\
\mathbb{P}(X_1=1,X_2=1,X_3=1) &= \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143}
\end{aligned}$$

3. Consider a sequence of independent Bernoulli trials, each of which is a success with probability p. Let X_1 be the number of failures preceding the first success, and let X_2 be the number of failures between the first two successes. Find the joint mass function of X_1 and X_2.

Solution: If $X_1 = i$ and $X_2 = j$, then the first i trials are failures, the $(i+1)$-th trial is success, and the next j trials are again failures, and the last $(i+j+2)$-th trial is success. Thus, we have $\mathbb{P}(X_1 = i, X_2 = j) = p^2(1-p)^{i+j}$.

4. The joint probability density function of X and Y is given by

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), 0 < x < 1, 0 < y < 2$$

and 0 otherwise.

(a) Verify that this is indeed a joint density function.

(b) Compute the density function of X.

(c) Find $\mathbb{P}(X > Y)$.

(d) Find $\mathbb{P}(Y > 1 \mid X < 1/2)$.

(e) Find $\mathbb{E}\,X$.

(f) Find $\mathbb{E}\,Y$.

Solution:

(a)

$$\int_0^1\int_0^2\left(x^2+\frac{xy}{2}\right)dydx = \int_0^1(2x^2+x)dx = \frac{7}{6}$$

(b)

$$f_X(x) = \frac{6}{7}(2x^2+x) \qquad \text{for } 0<x<1$$

(c)

$$\mathbb{P}(X>Y) = \int_0^1\int_0^x f(x,y)dydx = \frac{15}{56}$$

(d)

$$\mathbb{P}\left(Y > 1 \mid X < \frac{1}{2}\right) = \frac{\mathbb{P}\left(X < \frac{1}{2}, Y > 1\right)}{\mathbb{P}\left(X < \frac{1}{2}\right)} = \frac{\int_1^2 \int_0^{\frac{1}{2}} f(x,y)dxdy}{\int_0^{\frac{1}{2}} f_X(x)dx} = 0.65.$$

(e)

$$\mathbb{E}\,X = \int_0^1 x f_X(x)dx = \frac{5}{7}$$

(f)

$$\mathbb{E}\,Y = \int_0^2 y \int_0^1 f(x,y)dxdy = \frac{8}{7}$$

5. Let X, Y be jointly distributed with density function $f(x,y) = e^{-(x+y)}$ for $0 \leqslant x < \infty$, $0 \leqslant y < \infty$. Find (a) $\mathbb{P}(X < Y)$ and (b) $\mathbb{P}(X < a)$ for $a \in \mathbb{R}$.

Solution:

(a) Since $\mathbb{P}(X < Y) = \mathbb{P}(X \geqslant Y)$, we have $\mathbb{P}(X < Y) = \frac{1}{2}$.

(b) If $a < 0$, then $\mathbb{P}(X < a) = 0$. Let $a \geqslant 0$, then

$$\mathbb{P}(X < a) = \int_0^a \int_0^\infty e^{-(x+y)} dydx = 1 - e^{-a}.$$

6. A man and a woman agree to meet at a certain location about 12:30PM. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1PM, find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

Solution: Let X be uniform on $(-15, 15)$, and let Y be uniform on $(-30, 30)$. Nobody waits longer than five minutes if $|Y - X| < 5$.

$$\begin{aligned}\mathbb{P}(|Y - X| < 5) = \mathbb{P}(-5 < Y - X < 5) = \mathbb{P}(X - 5 < Y < X + 5)\\ = \int_{-15}^{15} \int_{x-5}^{x+5} \frac{1}{30 \cdot 60} dydx = \frac{30 \cdot 10}{30 \cdot 60} = \frac{1}{6}.\end{aligned}$$

The probability that the man arrives first is $\mathbb{P}(X < Y) = \frac{1}{2}$ by symmetry.

7. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} xe^{-(x+y)}, & x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether X and Y are independent.

Solution: Since $f(x,y) = f_X(x)f_Y(y)$, where $f_X(x) = xe^{-x}$ for $x > 0$, and $f_Y(y) = e^{-y}$ for $y > 0$ (0 otherwise), so that X and Y are independent.

8. The joint density function of X and Y is $f(x,y) = \begin{cases} x + y & \text{if } 0 < x < 1, 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$

(a) Are X and Y independent?

(b) Find the density function of X.

(c) Find $\mathbb{P}(X+Y<1)$.

(d) Find $\mathbb{E}\,X$.

(e) Find $\mathrm{Var}(X)$.

Solution: Let X and Y be jointly continuous with density function

$$f(x,y) = \begin{cases} x+y & \text{if } 0<x<1, 0<y<1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) X and Y are not independent, since $f(x,y)$ is clearly not a product of functions of x and y.

(b) $f_X(x) = \int_0^1 (x+y)dy = \left(xy + \frac{y^2}{2}\right)\Big|_0^1 = x + \frac{1}{2}$ for $0<x<1$.

(c) $\mathbb{P}(X+Y<1) = \int_0^1 \int_0^{1-y} (x+y)dxdy = \int_0^1 \left(\frac{(1-y)^2}{2} + y(1-y)\right) dy = \frac{1}{2}\int_0^1 (1-y^2)dy = \frac{1}{2}(1-\frac{1}{3}) = \frac{1}{3}$.

(d) $\mathbb{E}\,X = \int_0^1 x f_X(x)dx = \int_0^1 x(x+\frac{1}{2})dx = (x^3/3 + x^2/4)\Big|_0^1 = 7/12$.

(e) $\mathbb{E}\,X^2 = \int_0^1 x^2(x+\frac{1}{2})dx = 5/12$ and $\mathrm{Var}(X) = 5/12 - (7/12)^2 = 11/144$.

9. Let X and Y be jointly distributed with density function

$$f(x,y) = \begin{cases} 12xy(1-x), & 0<x<1, 0<y<1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Are X and Y independent?

(b) Find $\mathbb{E}\,X$.

(c) Find $\mathbb{E}\,Y$.

(d) Find $\mathrm{Var}(X)$.

(e) Find $\mathrm{Var}(Y)$.

Solution: First, compute $f_X(x) = \int_0^1 12xy(1-x)dy = 6x(1-x)$ and $f_Y(y) = \int_0^1 12xy(1-x)dy = 2y$.

(a) Clearly, $f(x,y) = f_X(x)f_Y(y)$, so that X and Y are independent.

(b) $\mathbb{E}\,X = \int_0^1 6x^2(1-x)dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}$.

(c) $\mathbb{E}\,Y = \int_0^1 2y^2 dy = \frac{2}{3}y^3|_0^1 = \frac{2}{3}$.

(d) First, find $\mathbb{E}\,X^2 = \int_0^1 6x^3(1-x)dx = \frac{3}{2}x^4 - \frac{6}{5}x^5|_0^1 = \frac{3}{10}$. Now, $\mathrm{Var}(X) = \mathbb{E}\,X^2 - EX^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

(e) First, find $\mathbb{E}\,Y^2 = \int_0^1 2y^3 dy = \frac{1}{2}y^4|_0^1 = \frac{1}{2}$. Now, $\mathrm{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.

10. If X_1 and X_2 are independent exponential random variables with respective parameters λ_1 and λ_2, find the distribution of $Z = X_1/X_2$. Also compute $\mathbb{P}(X_1 < X_2)$.

Solution: Let X_1, X_2 be exponential random variables with parameter λ_1, λ_2. Let $Z = X_1/X_2$. Note that $F_Z(a) = 0$ if $a \leqslant 0$. Compute $F_Z(a)$ for $a > 0$:

$$\begin{aligned} F_Z(a) = \mathbb{P}(Z \leqslant a) = \mathbb{P}(X_1 \leqslant aX_2) &= \lambda_1\lambda_2 \int_0^\infty \int_0^{ay} e^{-\lambda_1 x - \lambda_2 y} dx dy \\ &= \lambda_2 \int_0^\infty e^{-\lambda_2 y}(1 - e^{-\lambda_1 a y}) dy \\ &= 1 - \frac{\lambda_2}{\lambda_1 a + \lambda_2}, \end{aligned}$$

so that

$$f_Z(a) = \frac{d}{da} F(a) = \frac{\lambda_1 \lambda_2}{(a\lambda_1 + \lambda_2)^2}.$$

Finally, we have

$$\mathbb{P}(X_1 < X_2) = \mathbb{P}(Z < 1) = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$