Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.1 Systems of Linear Equations

Topics

We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

A Single Linear Equation

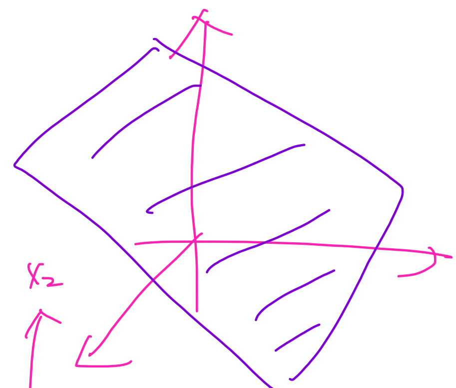
A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the **dimension**, or number of variables.

For example,

- $2x_1 + 4x_2 = 4$ is a line in two dimensions
- $3x_1 + 2x_2 + x_3 = 6$ is a plane in three dimensions



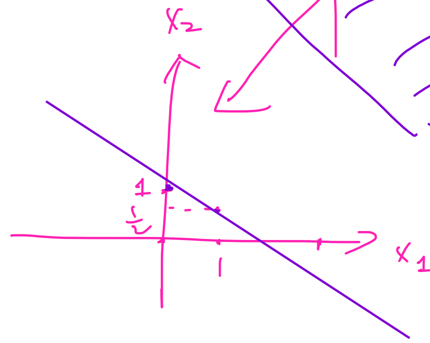
$(x_1, x_2) = (0, 0) \leftarrow$ Not Solution.

$(0, 1) \leftarrow$ Solution

$(1, \frac{1}{2})$

$\vdots$

$(x_1, 1 - \frac{1}{2}x_1)$



Systems of Linear Equations

When we have more than one linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\left\{\begin{array}{rrrcl} x_1 & +\,1.5x_2 & +\,\pi x_3 & = & 4 \\ 5x_1 & +\,0\cdot x_2 & +\,7x_3 & = & 5 \end{array}\right.$$

Definition: Solution to a Linear System

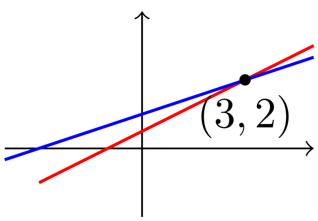
The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution** to the system.

A system can have a unique solution, no solution, or an infinite number of solutions.

Two Variables

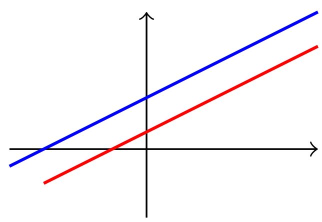
Consider the following systems. How are they different from each other?

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$



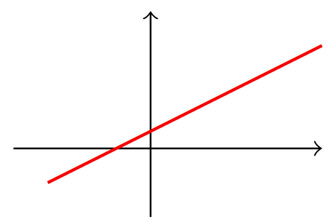
non-parallel lines

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



parallel lines

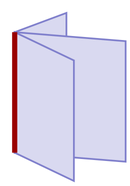
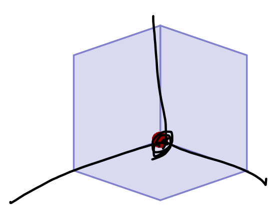
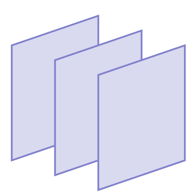
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

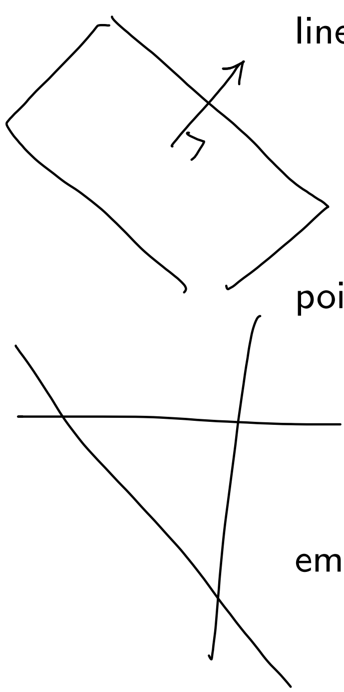


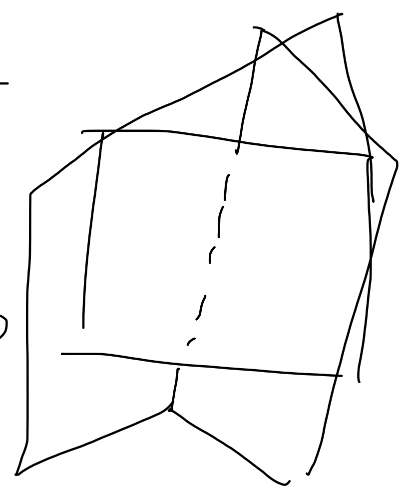
identical lines

Three-Dimensional Case

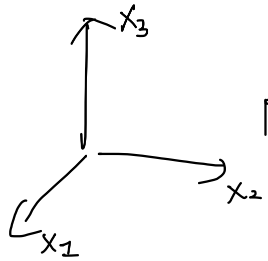
An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	sketch	number of solutions
line		Infinity.
point		1
empty		0





$\mathbb{R}^3 = \{ (x_1, x_2, x_3) : x_1, x_2, x_3$ are real numbers$\}$



Q: Any system has 2 solutions?

Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one row to another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

Example 1

Identify the solution to the linear system.

$$\begin{array}{rrrrl}
\overset{5}{x_1} & \overset{-10}{-2x_2} & \overset{5}{+x_3} & \overset{0}{=0} & \text{——— ①}\\
 & 2x_2 & -8x_3 & =8 & \text{——— ②}\\
5x_1 & & -5x_3 & =10 & \text{——— ③}
\end{array}$$

$$\begin{array}{rrrl}
10x_2 & -10x_3 & =10 & \text{——— ③} - 5\times ① = ④\\
x_2 & - x_3 & = 1 & \longrightarrow \frac{1}{10}\times ④ = ⑤\\
 & x_3 = & \bigcirc & \quad ② - 2\times ⑤
\end{array}$$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{rrrl} 1x_1 & -2x_2 & +x_3 & = 0 \\ 0 \cdot x_1 & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3.

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \xrightarrow[\text{Elementary Row operation}]{R_3 - 5 \times R_1 \to R_3} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right]$$

$$\xrightarrow{\frac{1}{10} \times R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$- - - \to$

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one Solution.

otherwise, inconsistent

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of row operations transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Fundamental Questions

Two questions that we will revisit many times throughout our course.

1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics
We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

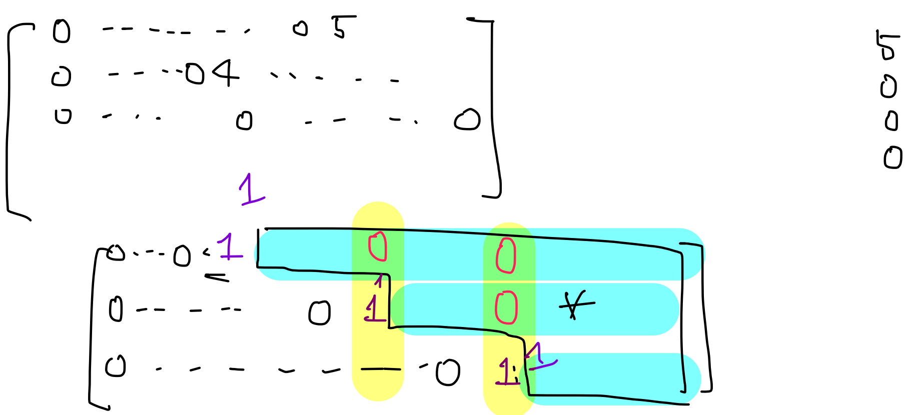
Definition: Echelon Form and RREF

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. All elements below a leading entry (if any) are zero.

A matrix in echelon form is in **reduced row echelon form** (RREF) if

1. All leading entries, if any, are equal to 1.
2. Leading entries are the only nonzero entry in their respective column.



Example of a Matrix in Echelon Form

$\blacksquare$ = non-zero number, $\quad *$ = any number

← Pivot position

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot column

Example 1

Which of the following are in RREF?

$$a)\quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \xrightarrow[\frac{1}{2}\times R_2 \to R_2]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad d)\quad \begin{bmatrix} 0 & 6 & 3 & 0 \end{bmatrix} \xrightarrow[\frac{1}{6}R_1 \to R_1]{} \begin{bmatrix} 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

$$b)\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Yes} \qquad e)\quad \begin{bmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{RREF.}$$

$$c)\quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

Example 2: Express the matrix in reduced row echelon form and identify the pivot columns.

$$\left[\begin{array}{ccc|c} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 \leftrightarrow R_2 \\ (-1)\cdot R_1 \to R_1}]{} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & -3 & -6 & 4 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_3 + 2\cdot R_1 \to R_3 \\ -\frac{1}{3} \times R_2 \to R_2}]{} \begin{bmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -\frac{4}{3} \\ 0 & 1 & 2 & -3 \end{bmatrix} \xrightarrow[R_1 - 2R_2 \to R_1]{} \begin{bmatrix} 1 & 0 & -3 & -\frac{1}{3} \\ 0 & 1 & 2 & -\frac{4}{3} \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

$-2 + 2\cdot 1 \mid -3 + 2\cdot 2 \mid 0 + 2\cdot 1 \mid 3 + 2\cdot(-3)$

$$\xrightarrow[R_3 - R_2 \to R_3]{} \begin{bmatrix} 1 & 0 & -3 & -\frac{1}{3} \\ 0 & 1 & 2 & -\frac{4}{3} \\ 0 & 0 & 0 & -\frac{5}{3} \end{bmatrix} \xrightarrow[\substack{-\frac{3}{5} R_3 \to R_3 \\ R_1 + \frac{1}{3} R_3 \to R_1 \\ R_2 + \frac{4}{3} R_3 \to R_2}]{} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \text{RREF.}$$

Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

Step 1a	Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row
Step 1b	Scale the 1st row so that its leading entry is equal to 1
Step 1c	Use row replacement so all entries below this 1 are 0
Step 2a	Swap the 2nd row with a lower one so that the leftmost nonzero entry below 1st row is in the 2nd row
etc.	...
	Now the matrix is in echelon form, with leading entries equal to 1.
Last step	Use row replacement so all entries above each leading entry are 0, starting from the right.

Basic And Free Variables

Consider the augmented matrix

$$\left[A \mid \vec{b}\right] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

$$\begin{cases} x_1 + 3x_2 + 7x_4 = 4 \\ x_3 + 4x_4 = 5 \\ x_5 = 6 \end{cases}$$

The leading one's are in first, third, and fifth columns. So:

- the pivot variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3, and x_5. basic
- The free variables are x_2 and x_4. **Any choice** of the free variables leads to a solution of the system.

Note that A does not have basic variables or free variables. Systems have variables.

$$\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 + 5x_2 = -7 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 3 & 5 & -7 \end{array}\right] \text{ Augmented matrix.}$$

$\downarrow$ row operations

$\left[\quad\quad\right]$ RREF.

$$\left(\begin{array}{cccccc|c} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\ & & & & & & 0 \\ & & & & & & \vdots \end{array}\right) \quad \text{RREF}$$

Existence and Uniqueness

$$0 = 0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 1$$

Theorem

A linear system is consistent if and only if (exactly when) the last column of the **augmented** matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no free variables.
2. infinitely many solutions that are parameterized by free variables.

Linear systems $\longleftrightarrow$ matrices

$\nwarrow\!\!\searrow$ $\quad$ $\nearrow\!\!\swarrow$

Geometry

$$\text{RREF} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right) \longrightarrow \begin{aligned} 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 &= 2 \\ x_2 &= 3 \\ x_3 &= 4 \end{aligned}$$

no non-pivot. = no free variable

not pivot.

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

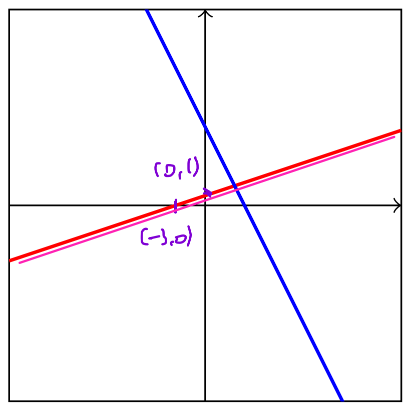
$(-3\ ,\ 0)$

$(0\ ,\ 1)$

$\vdots$

$$x - 3y = -3$$

$$2x + y = 8$$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

$\mathbb{R}^n$

$i^2 = -1$

$i \notin$

$x^2 \geq 0$

$\mathbb{R} = \{ x : x \text{ is real} \}$

1, 2, 3, ⋯ natural
0 } integers
−1, −2, ⋯

a/b ← rational #.

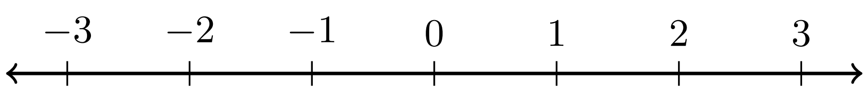
$\sqrt{2}$, π, ⋯ ← irr.

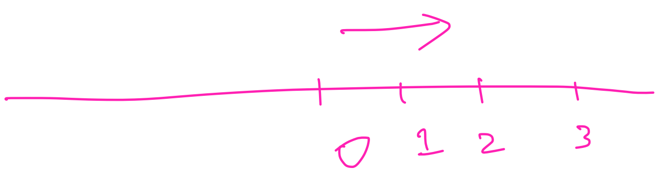
Recall that $\mathbb{R}$ denotes the collection of all real numbers.

Let n be a positive whole number. We define

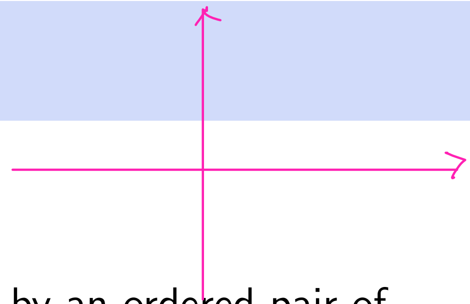
$$\mathbb{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.





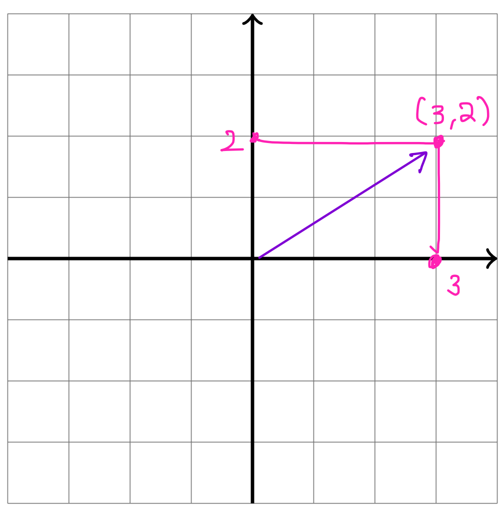
$\mathbb{R}^2$



Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. = an arrow from $(0,0)$ to $(3,2)$



= $\mathbb{R}^2$.

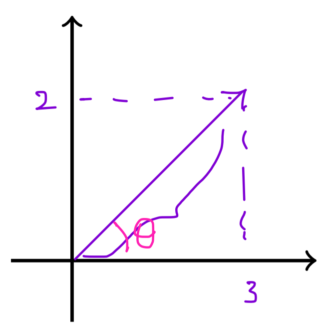
Vectors

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.

$= \sqrt{3^2 + 2^2}$

$= \sqrt{13}$.



For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \qquad \in \mathbb{R}^2$$

Vectors have the following properties.

1. **Scalar Multiple**:

   $c \in \mathbb{R}$

   $$\underline{c}\vec{u} = \begin{pmatrix} c \cdot u_1 \\ c \cdot u_2 \end{pmatrix}$$

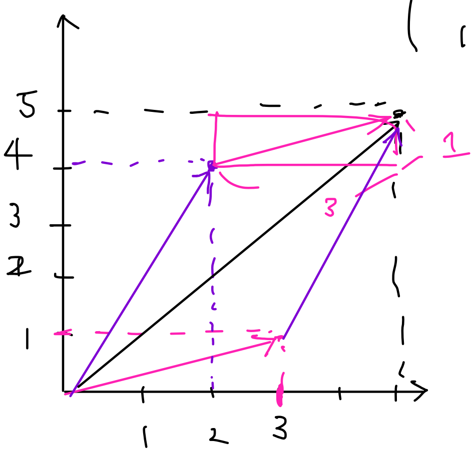
2. **Vector Addition**:

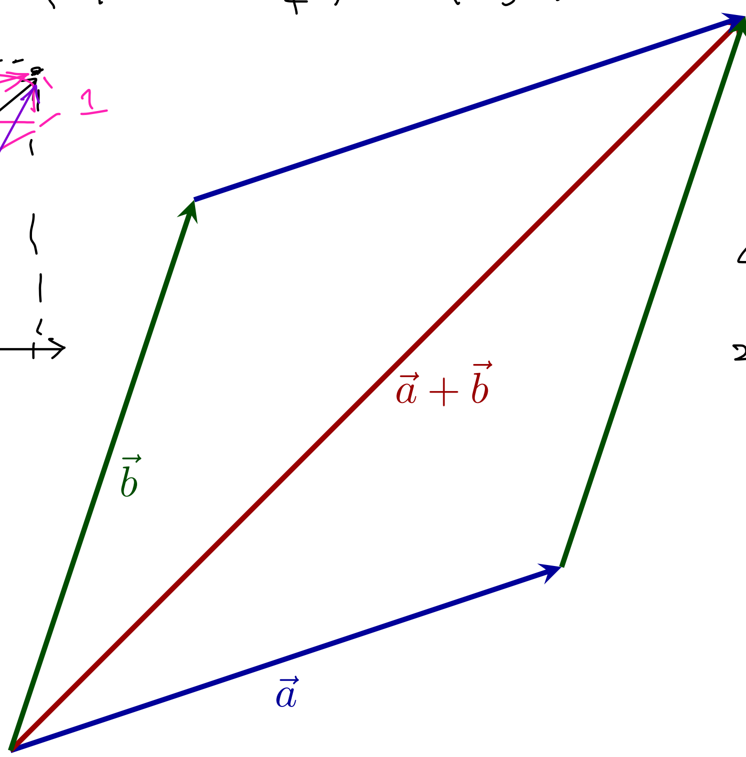
   $$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$$

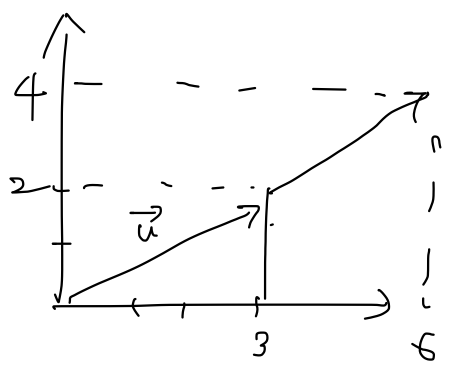
Note that vectors in higher dimensions have the same properties.

Parallelogram Rule for Vector Addition

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$







$$2 \cdot \vec{u} = \begin{pmatrix} 2 \cdot 3 \\ 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Linear Combinations and Span

$3 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Definition

1. Given vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p \in \mathbb{R}^n$, and scalars $\in \mathbb{R}$ $c_1, c_2, \ldots, c_p$, the vector below

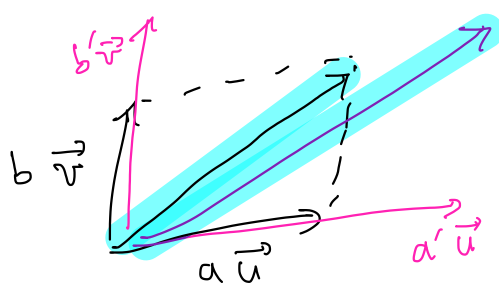
$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

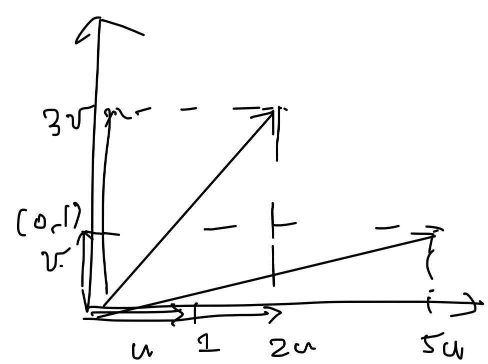
   is called a **linear combination of** $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ **with weights** $c_1, c_2, \ldots, c_p$.

2. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \vec{u}$ an $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \vec{v}$

$$= \left\{ a \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 4 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$





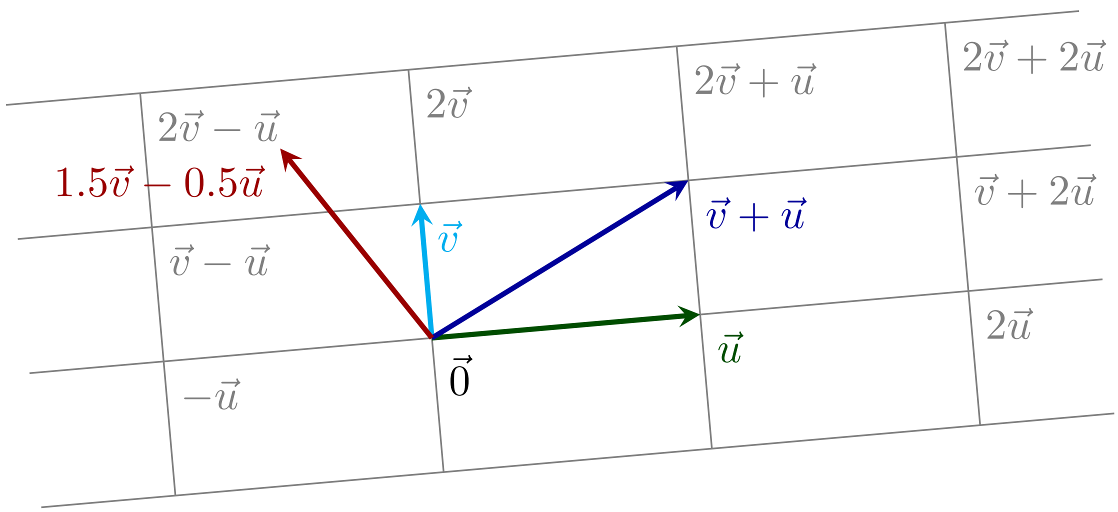
Span of $\vec{u}, \vec{v}$

$= \mathbb{R}^2$

Geometric Interpretation of Linear Combinations

$=$ they have different directions

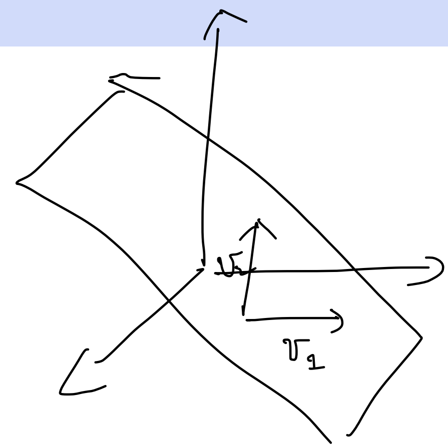
Note that any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$. In other words, any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors that are not multiples of each other.

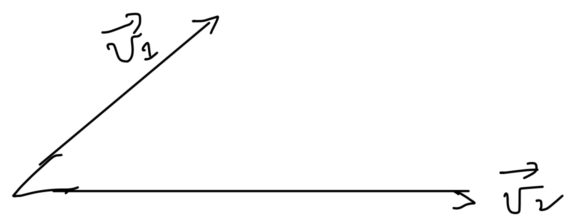


Example

Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$





Q: $\vec{y} \in$ Span of $\vec{v_1}$ and $\vec{v_2}$

$= \{ a \cdot \vec{v_1} + b\vec{v_2} \}$

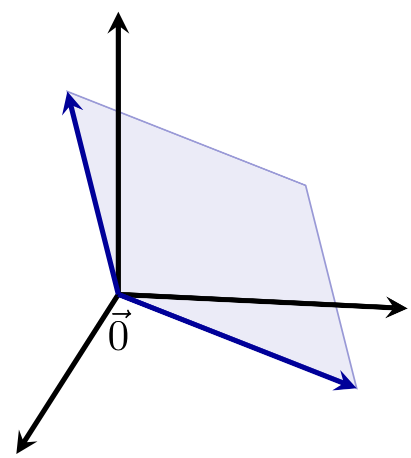
Can we find a, b s.t.

$\vec{y} = a\vec{v_1} + b\vec{v_2}$?

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.



Section 1.4 : The Matrix Equation

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

"Mathematics is the art of giving the same name to different things."
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

1.4 : Matrix Equation $A\vec{x} = \vec{b}$

Topics

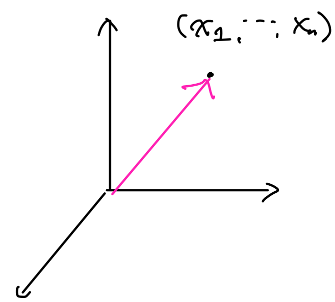
We will cover these topics in this section.

1. Matrix notation for systems of equations.
2. The matrix product $A\vec{x}$.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute matrix-vector products.
2. Express linear systems as vector equations and matrix equations.
3. Characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots.



Recall

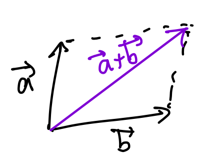
$$\mathbb{R}^n = \{ (x_1, x_2, \cdots, x_n) : x_1, \cdots, x_n \in \mathbb{R} \}$$

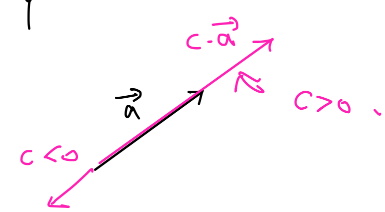
$$= \left\{ \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\text{vector}} \right\}$$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ vector

$c_1 \cdot \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p$: linear combination of $\vec{v}_1, \cdots, \vec{v}_p$

Span of $\vec{v}_1, \cdots, \vec{v}_p$ $= \{ c_1 \cdot \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_p \vec{v}_p : c_1, \cdots, c_p \in \mathbb{R} \}$





Notation

element $\downarrow$ $a \in A$ $\swarrow$ set

symbol	meaning
$\in$	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m\times n}$	the set of real-valued matrices with m rows and n columns

$\mathbb{R}^{2\times 3} \neq \mathbb{R}^6$ complex

Example: the notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.

$$\mathbb{R}^{2\times 3} = \{ \text{matrices with 2 rows \& 3 columns} \}$$

$$= \left\{ \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} : a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R} \right\}$$

$\in \mathbb{C}$

Ex $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1\cdot 1 + c_2\cdot 2 + c_3\cdot 0 \\ c_1\cdot 2 + c_2\cdot 0 + c_3\cdot 1 \\ c_1\cdot 0 + c_2(-3) + c_3\cdot 1 \end{bmatrix}$$

Linear Combinations

$$= \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Definition

$A \in \mathbb{R}^{m\times n}$

A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and $x \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of A:

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

linear combination of $\vec{a}_1, \ldots, \vec{a}_n$

Note that $A\vec{x}$ is in the span of the columns of A.

Example

The following product can be written as a linear combination of vectors:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix}}_{\mathbb{R}^{2\times 3}}\underbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}}_{\mathbb{R}^3 = \mathbb{R}^{3\times 1}} = \underbrace{4\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\cdot\begin{bmatrix} 0 \\ -3 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 3 \end{bmatrix}}_{\text{linear combination of } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}} \in \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right)$$

$$[a_1, \cdots, a_n]\begin{bmatrix} — \vec{x}_1 — \\ — \vec{x}_2 — \\ \vdots \\ — \vec{x}_n — \end{bmatrix}$$

Solution Sets

$\vec{a}_i \in \cancel{\mathbb{R}^n}$ or $\mathbb{R}^m$ or $\cancel{?}$

Theorem $A \in \mathbb{R}^{m \times n}$ $\quad A = \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$

If A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$, and $x \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$\underline{A\vec{x} = \vec{b}}$$

$$\begin{bmatrix} a_{11} & | & & | \\ a_{12} & \vec{a}_2 & \cdots & \vec{a}_n \\ \vdots & & & \\ a_{1m} & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

which as the same set of solutions as the set of linear equations with the augmented matrix

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Linear systems

Augmented matrices $\longleftrightarrow$ Vector Equations

Matrix Equations

??

Existence of Solutions

Theorem

The equation $A\vec{x} = \vec{b}$ *has a solution if and only if* $\vec{b}$ *is a linear combination of the columns of* A*.*

Example

Q: For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \underbrace{\vec{x}}_{= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Q: $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ { is a linear combination / belongs to the span } of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$

Q: Does

$$\begin{cases} x_1 + 3x_2 + 4x_3 = b_1 \\ 2x_1 + 8x_2 + 4x_3 = b_2 \\ 0 \cdot x_1 + x_2 - 2x_3 = b_3 \end{cases}$$

have a solution?

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\xrightarrow{R_2 - 2 \cdot R_1 \to R_2} \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2 \cdot b_1 \\ 0 & 1 & -2 & b_3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 1 & -2 & b_3 \\ 0 & 2 & -4 & b_2 - 2 \cdot b_1 \end{bmatrix}$$

$=0$

$=0$

$$\xrightarrow[R_3 - 2 \cdot R_2 \to R_3]{R_1 - 3 \cdot R_2 \to R_1} \begin{bmatrix} 1 & 0 & 10 & b_1 - 3 \cdot b_3 \\ 0 & 1 & -2 & b_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(b_2 - 2 \cdot b_1) - 2 \cdot b_3$

pivots

not pivot

pivot or not.

$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$

$$\underline{b_2 - 2 \cdot b_1 - 2 \cdot b_3} \begin{cases} = 0 & \Rightarrow \text{Not pivot} \Rightarrow \underline{\text{Consistant}} \\ \neq 0 & \Rightarrow \text{pivot} \Rightarrow \text{No solution} \end{cases}$$

The Row Vector Rule for Computing $A\vec{x}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Summary

We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. An augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

Row operations

RREF.

3. A vector equation:

$$\underbrace{x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{\text{a linear combination}} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \in \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$$

4. As a matrix equation:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Each representation gives us a different way to think about linear systems.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.5 : Solution Sets of Linear Systems

Topics

We will cover these topics in this section.

1. Homogeneous systems
2. Parametric **vector** forms of solutions to linear systems

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Express the solution set of a linear system in parametric vector form.
2. Provide a geometric interpretation to the solution set of a linear system.
3. Characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ \qquad\qquad\qquad\qquad\qquad = b_m \end{cases}$$

Homogeneous Systems

$\boxed{A \cdot \vec{x} = \underset{=0}{\vec{b}}}$ or $\boxed{x_1 \cdot \vec{a}_1 + x_2 \cdot \vec{a}_2 + \cdots + x_n \vec{a}_n = \vec{b}}$ 0

Definition

Linear systems of the form $\vec{b} = 0$ are **homogeneous**.

Linear systems of the form $\vec{b} \neq 0$ are **inhomogeneous**.

is always consistent.

$\vec{0}$ unique solution

Infinitely many solution

Because homogeneous systems always have the **trivial solution**, $\vec{x} = \vec{0}$, the interesting question is whether they have a unique / infinitely many solutions.

$$\frac{\left[\begin{array}{c|c} A & 0 \end{array}\right]}{\underbrace{[\,A\,]}}$$

Observation

$A\vec{x} = \vec{0}$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & -5 \\ 1 & 0 & -2 \end{bmatrix} \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & -7 & -7 \\ 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 - 3R_2 \to R_1 \\ R_3 + 3R_2 \to R_3}]{-\frac{1}{7} \times R_2 \to R_2} \begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}}_{\text{pivot}} & \underbrace{\begin{matrix} -2 \\ 1 \\ 0 \end{matrix}}_{\text{not pivot}} \end{bmatrix}$$

RREF

not pivot.

$\to x_3$ is a free variable.

$$\begin{cases} x_1 \qquad - 2x_3 = 0 \\ 1 \cdot x_2 + 1 \cdot x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \end{cases}$$

$$\text{Solution set} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \right\}$$

$$= \left\{ \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ c \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$$

Parametric Forms, Homogeneous Case

In the example on the previous slide we expressed the solution to a system using a vector equation. This is a **parametric form** of the solution.

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

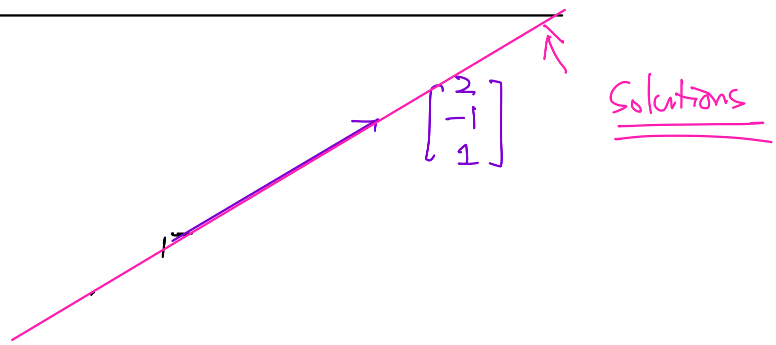
$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

a linear combination

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.

Ex Solution set $= \left\{ c \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$

$$x_1 \cdot \vec{a_1} + x_2 \cdot \vec{a_2} + x_3 \cdot \vec{a_3} = \vec{0}$$



Example 2 (non-homogeneous system)

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$x_1 + 3x_2 + x_3 = 9$$
$$2x_1 - x_2 - 5x_3 = 11$$
$$x_1 - 2x_3 = 6$$

(Note that the left-hand side is the same as Example 1).

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 2 & -1 & -5 & 11 \\ 1 & 0 & -2 & 6 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -7 & -7 & -7 \\ 0 & -3 & -3 & -3 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 - 3R_2 \to R_1 \\ R_3 + 3R_2 \to R_3}]{-\frac{1}{7} \times R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

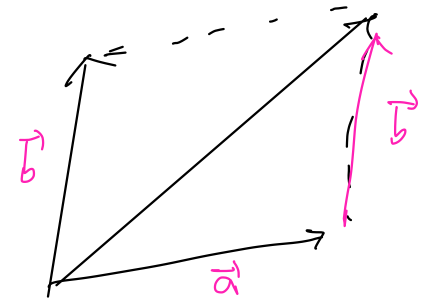
RREF.

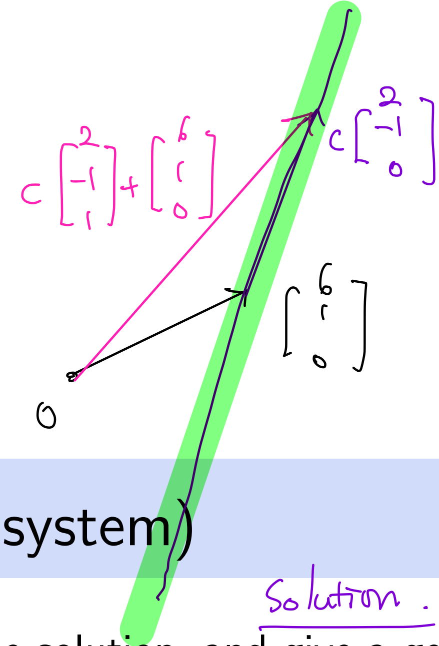
$$\begin{cases} x_1 - 2x_3 = 6 \\ x_2 + x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 = 2x_3 + 6 \\ x_2 = -x_3 + 1 \end{cases}$$

$$\begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Solution Set} = \left\{ \begin{bmatrix} 2x_3 + 6 \\ -x_3 + 1 \\ x_3 \end{bmatrix} : x_3 \in \mathbb{R} \right\} = \left\{ c \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\}$$





Example 2 (non-homogeneous system)

Solution.

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ 2x_1 - x_2 - 5x_3 &= 11 \\ x_1 - 2x_3 &= 6 \end{aligned}$$

(Note that the left-hand side is the same as Example 1).

Ex

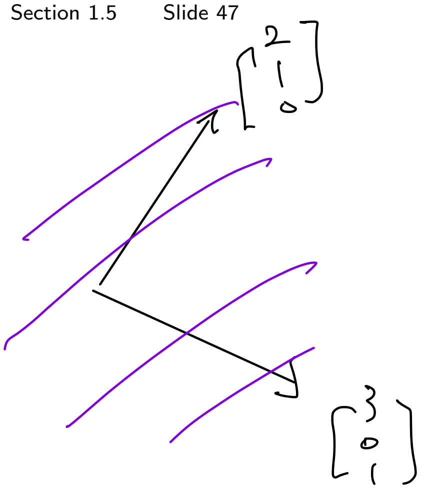
$$\cdots \longrightarrow \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 2x_2 + 3x_3$$

$$\text{Sol.} = \left\{ \begin{bmatrix} 2x_2+3x_3 \\ x_2 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

$$= \left\{ x_2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} : \text{''} \right\}$$



Section 1.7 : Linear Independence

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.7 : Linear Independence

Topics

We will cover these topics in this section.

- Linear independence
- Geometric interpretation of linearly independent vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a set of vectors and linear systems using the concept of linear independence.
2. Construct dependence relations between linearly dependent vectors.

Motivating Question

What is the smallest number of vectors needed in a parametric solution to a linear system?

Linear Independence

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

homogeneous

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

$$\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \vec{0}$$

has only the **trivial** solution. It is **linearly dependent** otherwise.

$c_1 = 0 = c_2 = \cdots = c_k$

In other words, $\{\vec{v}_1, \ldots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \ldots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Consider the vectors:

$$\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k$$

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$\underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k}_{\text{linear combination.}} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = V\vec{c} \overset{??}{=} \vec{0}$$

$V \cdot \vec{c} = \vec{0}$ implies $\vec{c} = \vec{0}$

Linear independence: There is NO non-zero solution $\vec{c}$

Linear dependence: There is a non-zero solution $\vec{c}$.

- We can find $\vec{c} \neq \vec{0}$ s.t. $V \cdot \vec{c} = \vec{0}$

Remark $\{\vec{v}_1 = \vec{v}_2, \vec{v}_3, \vec{v}_4, \cdots, \vec{v}_k\}$ linearly dep. why?

because $\underset{\sim}{1} \cdot \vec{v}_1 + (\underline{-1})\vec{v}_2 + 0 \cdot \vec{v}_3 + \cdots + 0 \cdot \vec{v}_k = \vec{0}$

$\underline{c_1}\vec{v}_1 + \cdots + \underline{c_k}\vec{v}_k = \vec{0}$ has non-zero solution

Example 1

For what values of h are the vectors linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \ \overset{\text{“}\vec{v}_2\text{”}}{\begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}}, \ \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix} = \vec{v}_3$$

$h \neq 1, -2$

$\Rightarrow$ linearly indep.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly independent

$\overset{\text{def}}{\Longleftrightarrow}$ $c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$ implies $c_1 = c_2 = c_3 = 0$.

$\Leftrightarrow$ $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$ has the unique solution.

$\Leftrightarrow$ $\begin{bmatrix} 1 & 1 & h \\ 1 & h & 1 \\ h & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}}$ $\begin{cases} 3 \text{ pivot columns} \\ \text{No non-pivot columns.} \end{cases}$

$\xrightarrow[R_3 - h \cdot R_1 \to R_3]{R_2 - R_1 \to R_2}$ $\begin{bmatrix} 1 & 1 & h \\ 0 & h-1 & 1-h \\ 0 & 1-h & 1-h^2 \end{bmatrix}$ $\quad h \neq 1$

$1^2 - h^2 = (1+h) \cdot (1-h)$

$\xrightarrow[\frac{1}{1-h} R_3 \to R_3]{\frac{1}{h-1} R_2 \to R_2}$ $\begin{bmatrix} 1 & 1 & h \\ 0 & 1 & -1 \\ 0 & 1 & \underbrace{\frac{1-h^2}{1-h}}_{= 1+h} \end{bmatrix}$

$\xrightarrow[R_3 - R_2 \to R_3]{R_1 - R_2 \to R_1}$ $\begin{bmatrix} 1 & 0 & h+1 \\ 0 & 1 & -1 \\ 0 & 0 & h+2 \end{bmatrix} \Rightarrow h+2 \neq 0$

$h = -2$ $\quad 1 \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Example 2 (One Vector)

Suppose $\vec{v} \in \mathbb{R}^n$. When is the set $\{\vec{v}\}$ linearly dependent?

$c \cdot \vec{v} = \vec{0}$

if there is $c \neq 0$ then $\{\vec{v}\}$ linearly dep.

Otherwise, linearly indep.

$\Rightarrow$ Case 1: $\vec{v} = \vec{0}$ $\Rightarrow$ linearly dep.

Case 2: $\vec{v} \neq \vec{0}$ $\Rightarrow$ $c = 0$ is the only solution. linearly indep.

Example 3 (Two Vectors)

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent? Provide a geometric interpretation.

linearly dep. If there are c_1, c_2 not all zero such that

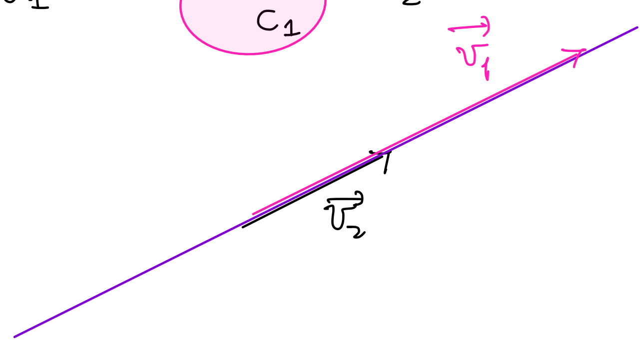
~~$c_1 = c_2 = 0$~~

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 = \vec{0}$$

Case 1: $c_1 \neq 0$

$$c_1 \vec{v}_1 = -c_2 \cdot \vec{v}_2$$

$$\vec{v}_1 = -\frac{c_2}{c_1} \cdot \vec{v}_2$$



Case 2: $c_2 \neq 0$

$$c_2 \vec{v}_2 = -c_1 \cdot \vec{v}_1$$

$$\vec{v}_2 = -\frac{c_1}{c_2} \cdot \vec{v}_1$$

Two Theorems

Fact 1. Suppose $\vec{v}_1, \ldots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

$$n \text{ rows} \left\{ \overbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}}^{k \text{ columns}} \right.$$

\# of maximum possible pivot columns

= \# of rows = n

< \# of columns

Fact 2. If any one or more of $\vec{v}_1, \ldots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

$$\underset{=1}{c_1} \cdot \underset{=0}{\vec{v}_1} + \underset{=0}{c_2}\vec{v}_2 + \cdots + \underset{=0}{c_k}\vec{v}_k = \vec{0}$$

$$1 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.8 : An Introduction to Linear Transforms

Topics

We will cover these topics in this section.

1. The definition of a linear transformation.
2. The interpretation of matrix multiplication as a linear transformation.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear).
2. Characterize linear transforms using the concepts of
   - existence and uniqueness
   - domain, co-domain and range

From Matrices to Functions

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(\vec{v}) = A \cdot \vec{v}$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$= \begin{bmatrix} 1 \cdot v_1 + 2 \cdot v_2 \\ 3 \cdot v_1 + 4 \cdot v_2 \end{bmatrix}$

$\in \mathbb{R}^2$

Let A be an $m \times n$ matrix. We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{v}$$

This is called a **matrix transformation**.

- The **domain** of T is $\mathbb{R}^n$.
- The **co-domain** or **target** of T is $\mathbb{R}^m$.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T
- The set of all possible images $T(\vec{x})$ is the **range**.

This gives us **another** interpretation of $A\vec{x} = \vec{b}$:

- set of equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$T(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \begin{bmatrix} 1 - 2 \\ 3 - 4 \end{bmatrix}$

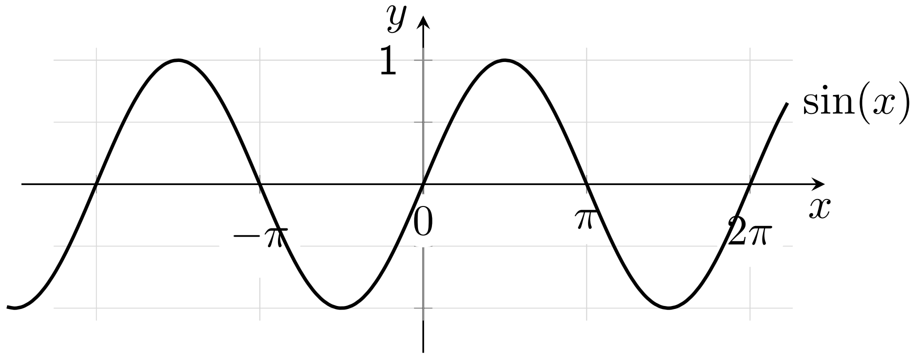
$= \underbrace{\begin{pmatrix} -1 \\ -1 \end{pmatrix}}_{\text{the image of } \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

Functions from Calculus

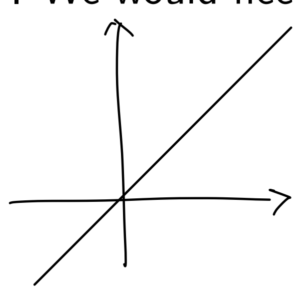
Many of the functions we know have **domain** and **codomain** $\mathbb{R}$.We can express the **rule** that defines the function $\sin$ this way:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph, whose horizontal axis is the **domain**, and the vertical axis is the **codomain**.



This is ok when the domain and codomain are $\mathbb{R}$. It's hard to do when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. We would need five dimensions to draw that graph.



Example 1 $\quad A = \mathbb{R}^{3\times 2} \qquad T: \mathbb{R}^2 \to \mathbb{R}^3$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$.

$T(\underline{\vec{x}}) = A \cdot \vec{x}$

$\vec{x} \in \mathbb{R}^2, \quad T(\vec{x}) \in \mathbb{R}^3$

a) Compute $T(\vec{u})$.

$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad T(\vec{u}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} u_1 + u_2 \\ u_2 \\ u_1 + u_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 7 \end{bmatrix}$

$u_1 = 3, \quad u_2 = 4$

b) Calculate $\vec{v} \in \mathbb{R}^2$ so that $\underline{T(\vec{v}) = \vec{b}}$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow T(\vec{v}) = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 1 & 1 & 7 \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{} \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 - R_2 \to R_1]{} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = 2, \quad v_2 = 5.$

c) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

pivot.

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$

Example 1

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$.

a) Compute $T(\vec{u})$.

b) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b}$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underline{\underline{v_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq}}$

$T : \mathbb{R}^n \to \mathbb{R}^m$

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

Image of Sum = Sum of Images

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$. ← Image of scalar multiple = Scalar multiple of Image.

So if T is linear, then

$$T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$$

Image of linear Combi. = linear Combi. of Images

This is called the **principle of superposition**. The idea is that if we know $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, then we know every $T(\vec{v})$.

Fact: Every matrix transformation T_A is linear.

$$T_A(\vec{x}) = A \cdot \vec{x}$$

$$T_A(\vec{u} + \vec{v}) = A \cdot (\vec{u} + \vec{v}) = A \cdot \vec{u} + A\vec{v}$$

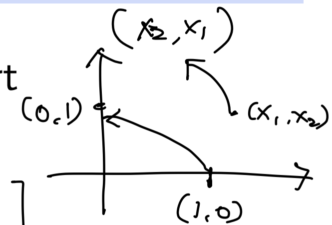
$$= T_A(\vec{u}) + T_A(\vec{v})$$

$$T_A(c \cdot \vec{u}) = A \cdot (c \cdot \vec{u})$$

$$= c \cdot (A\vec{u}) = c \cdot T_A(\vec{u})$$

Example 2

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.



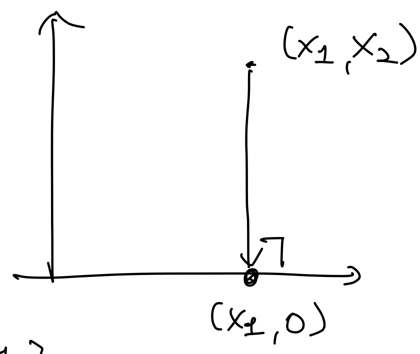
1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$T_A\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

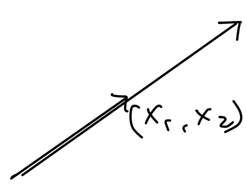
2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

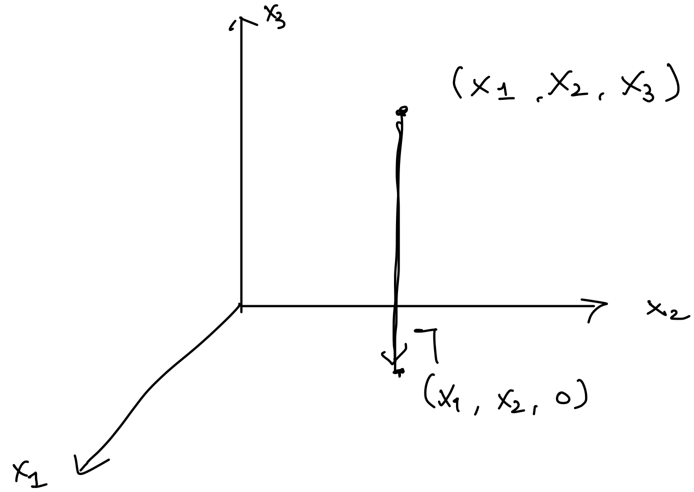


Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$



b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 4

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies

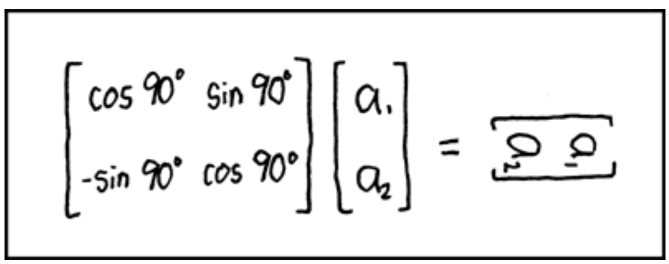
$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

What is the matrix that represents T?

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

1.9 : Matrix of a Linear Transformation

Topics

We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

Definition: The Standard Vectors

The **standard vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}}_{= A} \vec{e}_2 = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}}_{2^{nd} \text{ column of } A}$$

In general, $A \in \mathbb{R}^{m \times n}$

$$A \cdot \vec{e}_k \ (\in \mathbb{R}^m)$$

$\vec{e}_k \in \mathbb{R}^n$

$$A \cdot \vec{e}_k = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots \overset{\vec{a}_k}{\vee} \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th} \text{ entry}$$

$$= \vec{a}_k$$

The Standard Matrix

$\cdot\ T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$

def ? $\cdot\ T(c\cdot\vec{u}) = c\cdot T(\vec{u})$

for all $\vec{u}, \vec{v}, c$

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is the **standard matrix** for a linear transformation.

$$T(c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k)$$
$$= c_1 T(\vec{u}_1) + \cdots + c_k T(\vec{u}_k)$$

For a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n$

$$= \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{bmatrix}$$

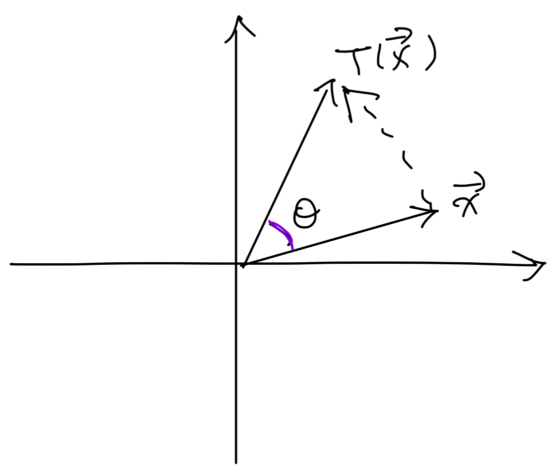
$$T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n)$$
$$= x_1 \underline{T(\vec{e}_1)} + \cdots + x_n \underline{T(\vec{e}_n)} = \overbrace{\begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}}^{A} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Rotations

Example 1

What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

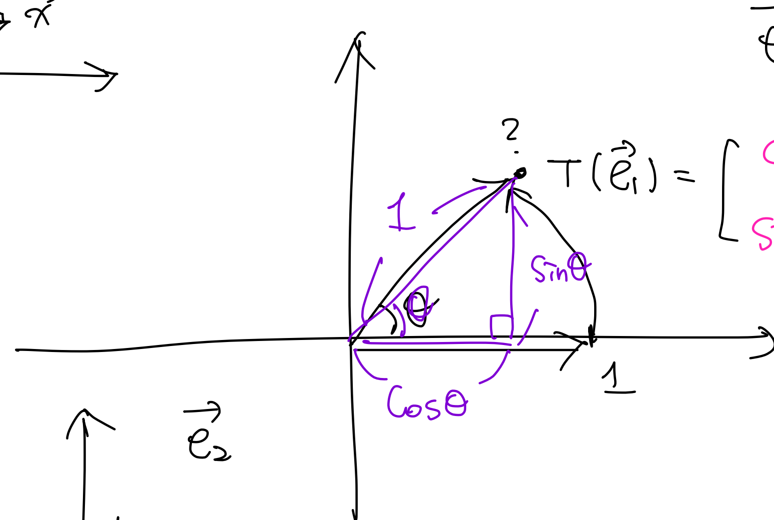
$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$



$$T(\vec{x}) = A \cdot \vec{x}$$

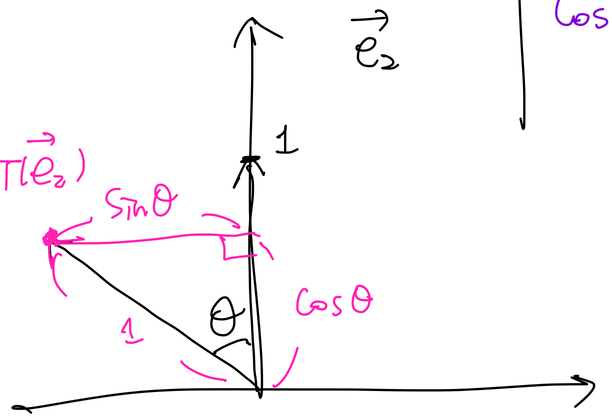
$$= \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \vec{x}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$\vec{e}_2$



$$= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

$\theta = 45^\circ = \frac{\pi}{4}$: $\cos(\theta) = \frac{\sqrt{2}}{2} = \sin\theta = \frac{1}{\sqrt{2}}$

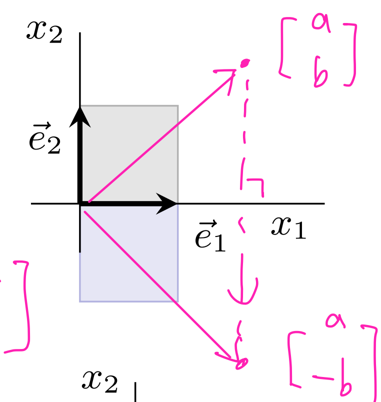
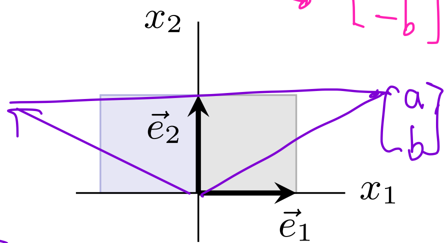
$$T(\vec{x}) = \begin{bmatrix} \frac{1}{\sqrt{2}}(x_1 - x_2) \\ \frac{1}{\sqrt{2}}(x_1 + x_2) \end{bmatrix}$$

Exercise : $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \dots$

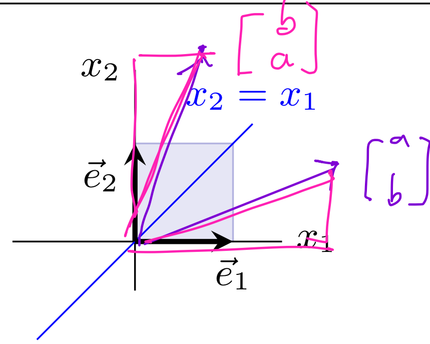
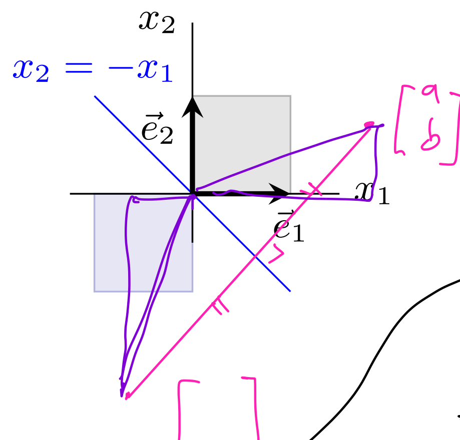
Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, . . .)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

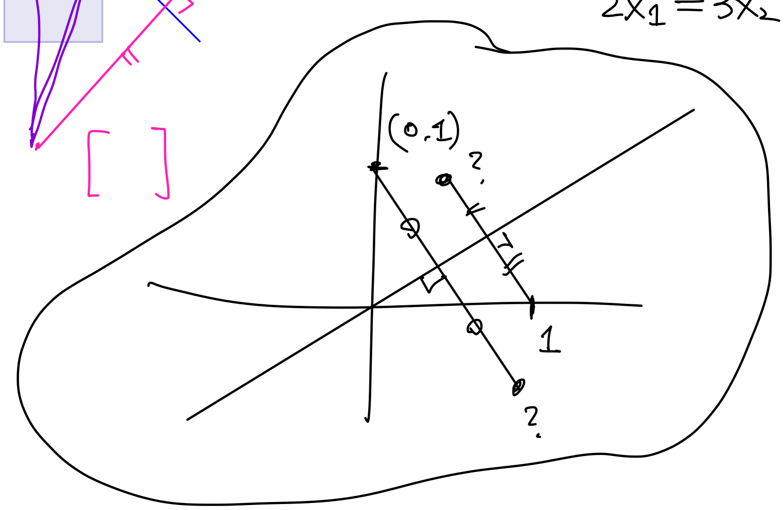
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1−axis $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection through x_2−axis $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

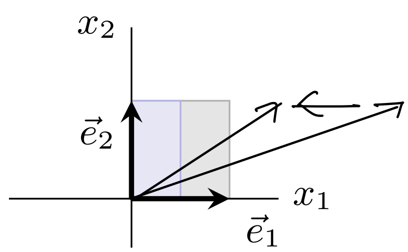
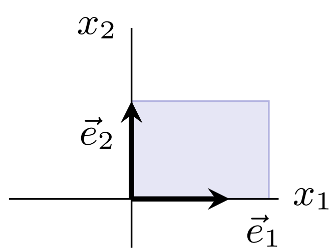
Two Dimensional Examples: Reflections

transformation	**image of unit square**	**standard matrix**
reflection through $x_2 = x_1$ $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ a \end{bmatrix}$ $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$ $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} -b \\ -a \end{bmatrix}$ $= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$2x_1 = 3x_2$



Two Dimensional Examples: Contractions and Expansions

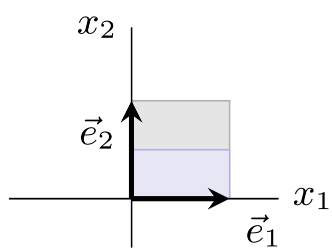
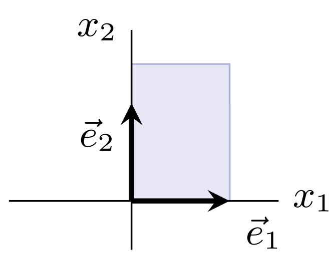
transformation	image of unit square	standard matrix
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. $\lvert k \rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

Ex

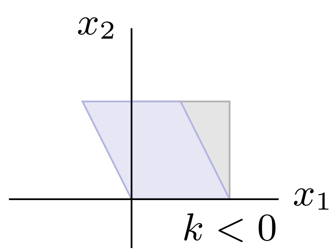
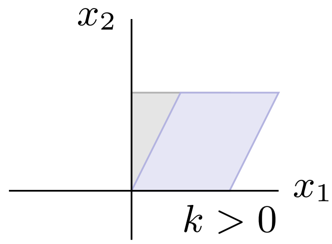
$$T(\vec{x}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \cdot x_1 \\ x_2 \end{bmatrix}$$

Two Dimensional Examples: Contractions and Expansions

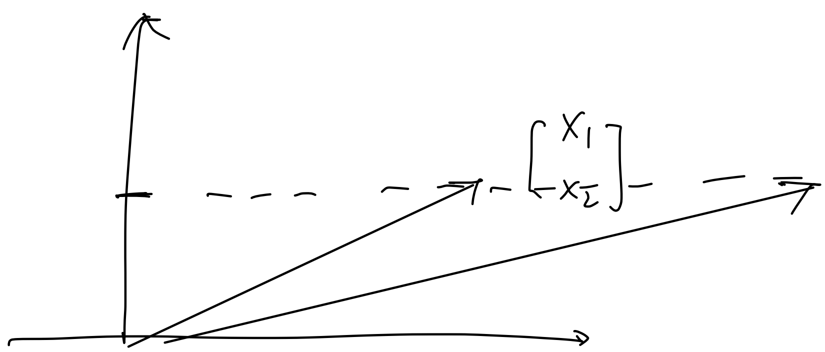
transformation	**image of unit square**	**standard matrix**
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k \rvert < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

Two Dimensional Examples: Shears

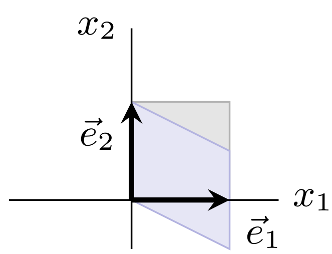
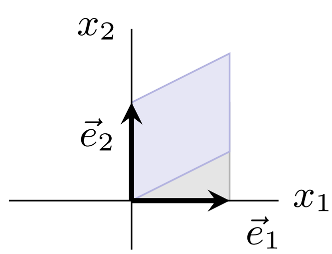
transformation	image of unit square	standard matrix
Horizontal Shear(left)		$\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$, $k < 0$
Horizontal Shear(right)		$\begin{pmatrix}1 & k\\ 0 & 1\end{pmatrix}$, $k > 0$

Ex

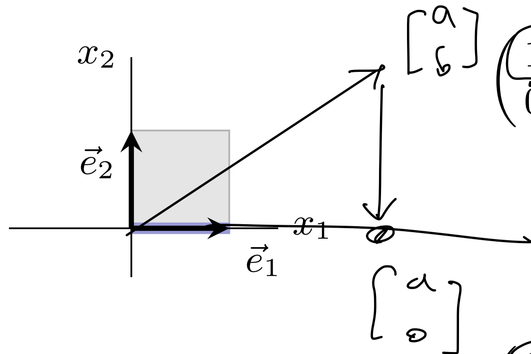
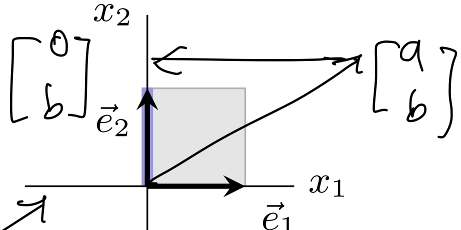
$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}x_1 + \underline{\underline{2}}x_2\\ x_2\end{pmatrix}$$

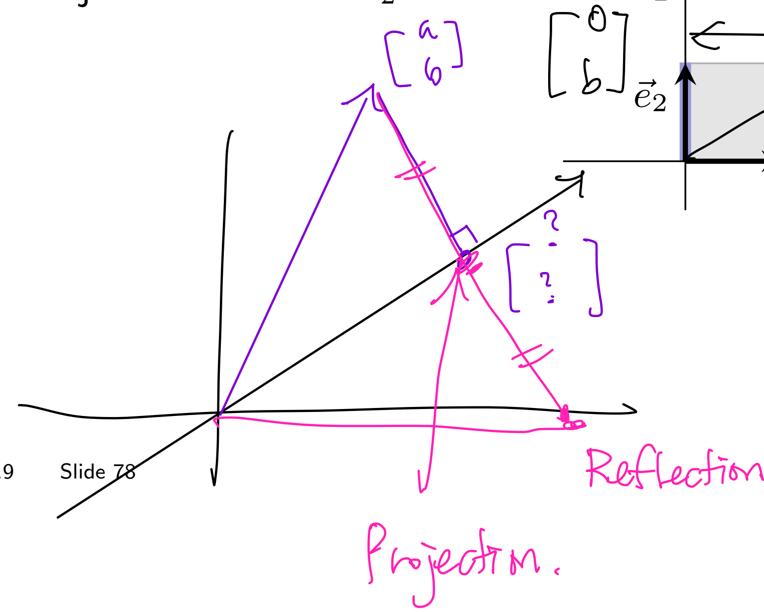


Two Dimensional Examples: Shears

transformation	**image of unit square**	**standard matrix**
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Projections

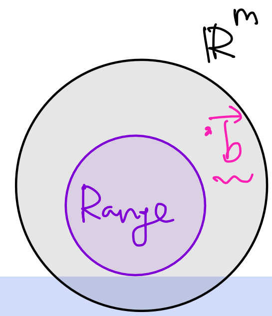
transformation	**image of unit square**	**standard matrix**
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$



Recall $T : \mathbb{R}^n \to \mathbb{R}^m$ linear if $\begin{cases} T(\vec u + \vec v) = T(\vec u) + T(\vec v) \\ T(c \cdot \vec u) = c \cdot T(\vec u) \end{cases}$

$\Rightarrow \quad T(\vec x) = A \cdot \vec x, \quad A \in \mathbb{R}^{m \times n}$

$A = [T(\vec e_1) \ \cdots \ T(\vec e_n)]$



Onto

$T : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^m}_{\text{Codomain}}$

Range of T = {all images} $= \{T(\vec x) : \vec x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

$=$ or $\subset$

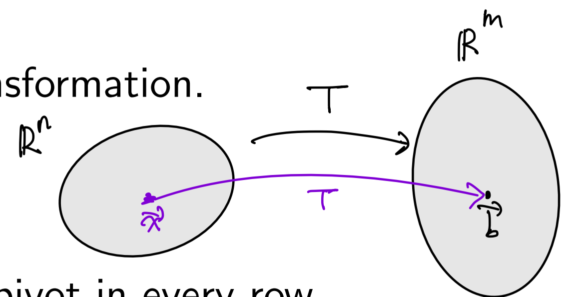
Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec b \in \underbrace{\mathbb{R}^m}_{\text{Codomain}}$ there is a $\vec x \in \mathbb{R}^n$ so that $\underline{T(\vec x)} = \vec b$.

Onto is an **existence property:** for any $\vec b \in \mathbb{R}^m$, $A\vec x = \vec b$ has a solution.

Examples

- A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.



Useful Fact

T is onto if and only if its standard matrix has a pivot in every row.

T is onto $\Leftrightarrow \forall \vec b \in \mathbb{R}^m, \ \exists \vec x$ s.t. $T(\vec x) = \vec b$

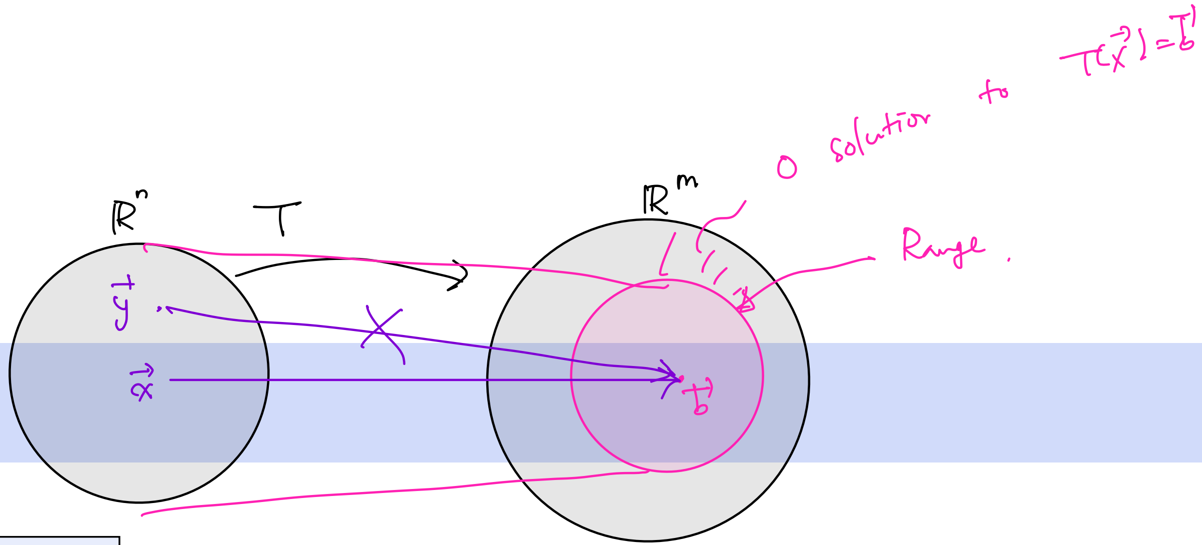
$\Leftrightarrow \forall \vec b \in \mathbb{R}^m, \ A\vec x = \vec b$ has a solution (is consistent).

$\Leftrightarrow \forall \vec b, \ [\, A \mid \vec b \,]$ the last column is NOT PIVOT

$\Leftrightarrow m \left\{ \overbrace{[\quad A \quad]}^{n} \right.$ Every ROW has leading entry

Note If T is onto, $m \leq n$. $T : \mathbb{R}^{\underline{n}} \to \mathbb{R}^{\underline{m}}$

One-to-One



Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.

Examples

- A rotation on the plane is a one-to-one linear transformation.
- A projection in the plane is not one-to-one.

Useful Facts

- T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if the standard matrix A of T has no free variables.

general

T is 1-1 $\iff$ If $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$

(for linear) $\iff$ If $T(\vec{x}) = \vec{0}$ then $\vec{x} = \vec{0}$

$\iff$ $A\vec{x} = 0$ has the only trivial solution

$\Leftrightarrow$ $\begin{bmatrix} A \end{bmatrix}$ has no non-pivot columns

Every Column is pivot

Columns of A is linearly indep.

Note T is 1–1 $\Rightarrow$ $m \geq n$

Example

$\underline{m \geq n}$ & $\underline{m \leq n}$

1–1 onto

$\Rightarrow$ $m = n$

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why**.

a) A is a 2×3 standard matrix for a one-to-one linear transform.

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$$ NP

Every column is pivot

b) B is a 3×2 standard matrix for an onto linear transform.

$$B = \begin{pmatrix} 1 & \\ & \\ & \end{pmatrix}$$ NP.

c) C is a 3×3 standard matrix of a linear transform that is one-to-one and onto.

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto.
2. The matrix A has columns which span $\mathbb{R}^m$.
3. The matrix A has m pivotal columns.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

Additional Examples

1. Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.
2. Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

   Is T one-to-one? Is T onto?

Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Definitions: Zero and Identity Matrices

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

Sums and Scalar Multiples

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A + B$ are $a_{i,j} + b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

For example, if

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + c \begin{bmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{bmatrix}$$

What are the values of c and k?

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

1. $A + 0_{m\times n} = A$
2. $(A+B)+C = A+(B+C)$
3. $r(A+B) = rA + rB$
4. $(r+s)A = rA + sA$
5. $r(sA) = (rs)A$

Matrix Multiplication

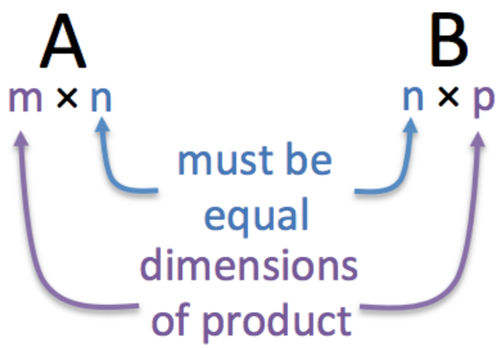
$\mathbb{R}^{m \times n} \times \mathbb{R}^{n \times p} = \mathbb{R}^{m \times p}$

Definition

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix}$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

Row Column Method

If $A \in \mathbb{R}^{m\times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n\times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

$$\begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{pmatrix} \begin{pmatrix} \vec{b}_1, \cdots, \vec{b}_p \end{pmatrix} = \begin{pmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \cdots & \vec{a}_1\cdot\vec{b}_p \\ \vec{a}_2\cdot\vec{b}_1 & \ddots & & \\ \vdots & & \ddots & \\ \vec{a}_m\cdot\vec{b}_1 & & & \vec{a}_m\cdot\vec{b}_p \end{pmatrix}$$

Example

Compute the following using the row-column method.

$$C = AB = \begin{array}{c} \vec{a}_1 \\ \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \\ \vec{a}_2 \end{array} \begin{array}{c} \begin{array}{ccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array} \\ \begin{pmatrix} 3 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix} \end{array}$$

$$= \begin{pmatrix} \vec{a}_1\cdot\vec{b}_1 & \vec{a}_1\cdot\vec{b}_2 & \vec{a}_1\cdot\vec{b}_3 \\ \vec{a}_2\cdot b_1 & \vec{a}_2\cdot\vec{b}_2 & \vec{a}_2\cdot\vec{b}_3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 2 \\ -1 & -5 & -5 \end{pmatrix}$$

Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

$I_m \in \mathbb{R}^{m\times m}$

$I_m = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots\ 1 \end{pmatrix}$

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B + C) = AB + AC$
3. (Right Distributive) $\cdots$ $(A+B)\cdot C = A\cdot C + B\cdot C$
4. (Identity for matrix multiplication) $I_m A = AI_n = A$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

- $A \in \mathbb{R}^{m\times n}$ $\quad B \in \mathbb{R}^{n\times p}$ $\quad A\cdot B \in \mathbb{R}^{m\times p}$

  $\underset{\mathbb{R}^{n\times \underline{p}}}{B} \cdot \underset{\mathbb{R}^{\underline{m}\times n}}{A}$ does not make sense

- If $m = p$ $\quad A\cdot B \in \mathbb{R}^{m\times m}$

  $B\cdot A \in \mathbb{R}^{n\times n}$

- If $m = n = p$ $\quad A\cdot B,\ B\cdot A \in \mathbb{R}^{m\times m}$

  $A\cdot B \neq B\cdot A$ in general

  $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

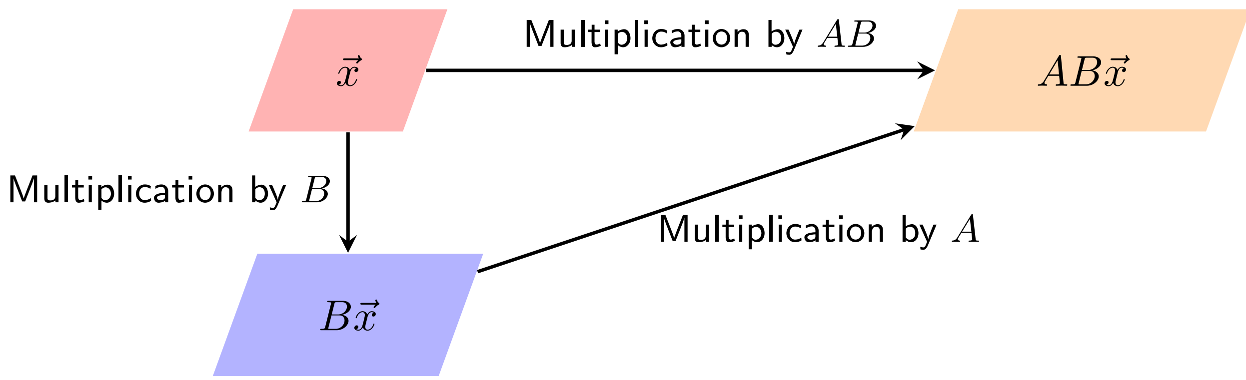
  $AB = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq B\cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix B that is non-commutative with A.

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$\mathbb{R}^{2\times 5}$ $\qquad\qquad$ $\mathbb{R}^{5\times 2}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 0 \\ 3 & 0 \\ 4 & 2 \\ 5 & 0 \end{bmatrix}$$

Properties of the Matrix Transpose

1. $(A^T)^T = A$.
2. $(A \pm B)^T = A^T + B^T$
3. $(rA)^T = r \cdot A^T$

because component wise operations

4. $(AB)^T = B^T \cdot A^T$

row

column

$A \in \mathbb{R}^{m\times n}$ $\quad B \in \mathbb{R}^{n\times p}$

$A^T \in \mathbb{R}^{n\times m}$, $B^T \in \mathbb{R}^{p\times n}$

Matrix Powers

For any $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = \underbrace{AA\dots A}_{k \text{ times}}$$

Example: Compute C^8.

diagonal matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

row

$$C^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$C^3 = C \cdot C^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 2^3 \end{bmatrix}$$

See pattern

$\vdots$

$$C^k = \begin{bmatrix} 1^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 2^k \end{bmatrix}$$

Example

Define

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of these operations are defined, and what are the dimensions of the result?

1. $A + 3C$

2. $A(AB)^T$

3. $A + ABCB^T$

Additional Examples

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$.

2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$.

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

- $a, b, c \in \mathbb{R}$, $a \cdot b = a \cdot c \Rightarrow b = c$?

  Not true because $a = 0$

- Question : A, B, C : matrices, $AB = AC$

  What condition of A makes $B = C$?

Topics and Objectives

Topics

We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

For this section, $A \in \mathbb{R}^{n \times n}$.

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

The Inverse of A

$$AC = CA = I_n. = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Identity

If there is, we write $C = A^{-1}$.

Ex $A \in \mathbb{R}^{1\times 1}$ $A = [a]$ $A^{-1} = ?$

If $a \neq 0$, $A^{-1} = [\frac{1}{a}]$

If $a = 0$, A is NOT invertible.

$$\begin{cases} a x_1 + b x_2 = 1 \\ a y_1 + b y_2 = 0 \\ c x_1 + d x_2 = 0 \\ c y_1 + d y_2 = 1 \end{cases}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Find $C \in \mathbb{R}^{2\times 2}$ s.t. $A \cdot C = C \cdot A = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$C = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$A \cdot C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ cx_1 + dx_2 & cy_1 + dy_2 \end{bmatrix}$

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular (invertible) if and only if $ad - bc \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \boxed{\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

Example

State the inverse of the matrix below.

$$A^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

$ad - bc = 2 \cdot (-7) - 5 \cdot (-3)$

$= -14 + 15 = 1 \neq 0$

$\therefore$ A is invertible

The Matrix Inverse

Theorem

is non-singular
is invertible

$A \in \mathbb{R}^{n \times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Note: Suppose $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.

$T(\vec{x}) = A\vec{x}$ is onto. 1-1

$$A \longrightarrow \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \vdots & & 1 \end{pmatrix} = I_n$$

Example

Solve the linear system.

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + 4x_2 &= 7 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

$$\vec{b} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$A^{-1}(A\vec{x}) = A^{-1} \cdot \vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\underbrace{A^{-1}A}_{= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}) \cdot \vec{x} = A^{-1} \cdot \vec{b}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3 \cdot 6 - 4 \cdot 5}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

$$\vec{x} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = -\frac{7}{2}\begin{bmatrix} 6-4 \\ -5+3 \end{bmatrix} = -\frac{7}{2}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ +7 \end{bmatrix}.$$

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

$A \cdot A^{-1} = I$

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

$\left(\because (A \cdot A^{-1})^T = (I)^T \right)$

$\underline{(A^{-1})^T \cdot A^T = I}$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

$\because (A \cdot B) \cdot C = \underline{\underline{I}}$

$(A \cdot \underbrace{B) \cdot (B^{-1}}_{= I} \cdot \boxed{}_{\uparrow A^{-1}}) = A \cdot \boxed{A^{-1}} = I$

An Algorithm for Computing A^{-1}

$A \cdot C = I$

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \,|\, I_n)$
2. If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$\begin{gathered} A\vec{x}_1 = \vec{e}_1 \\ A\vec{x}_2 = \vec{e}_2 \\ \vdots \\ A\vec{x}_n = \vec{e}_n \end{gathered}$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Over the next few slides we explore another explanation for how our algorithm works. This other explanation uses elementary matrices.

Recall $A \in \mathbb{R}^{n \times n}$ is invertible (non-singular) if there exists $C \in \mathbb{R}^{n \times n}$ such that $AC = CA = I_n$

$(AC = I)$ $(CA = I)$

$C = A^{-1}$

$[\, A \mid I_n \,] \xrightarrow[\text{row operations}]{} \cdots \longrightarrow [\, I_n \mid A^{-1} \,]$

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + 2 \cdot R_1 \to R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 + 2R_1 \to R_2]{} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Suppose

$$E \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

$$[\,A \mid I_n\,] \xrightarrow[(1)]{} [\,E_1 A \mid E_1\,] \xrightarrow[(2)]{} \cdots \xrightarrow[(k)]{} [\,I_n \mid A^{-1}\,]$$

$$I_n \xrightarrow{(1)} E_1 \qquad E_2 \qquad \cdots \qquad E_k$$

$$[\,E_2 E_1 A \mid E_2 E_1\,] \cdots$$

$$\xrightarrow[(k)]{} [\,\overbrace{E_k E_{k-1} \cdots E_2 E_1 A}^{I_n} \mid \underbrace{E_k E_{k-1} \cdots E_1}_{= A^{-1}}\,]$$

$A^{-1} = E_k E_{k-1} \cdots E_2 E_1$.

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k \cdots E_3 E_2 E_1) A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

Theorem

Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$\vec{x} = \underline{A^{-1}(A\vec{x})} = A^{-1}(\vec{b}) = \underline{A^{-1}\cdot\vec{b}}$$

  We would calculate A^{-1} and then:

  Finding inverse = Solving n systems.

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

Section 2.3 : Invertible Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

Topics and Objectives

Topics

We will cover these topics in this section.

1. The invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize the invertibility of a matrix using the Invertible Matrix Theorem.
2. Construct and give examples of matrices that are/are not invertible.

Motivating Question

When is a square matrix invertible? Let me count the ways!

The Invertible Matrix Theorem (IMT)

Invertible matrices enjoy a rich set of equivalent descriptions.

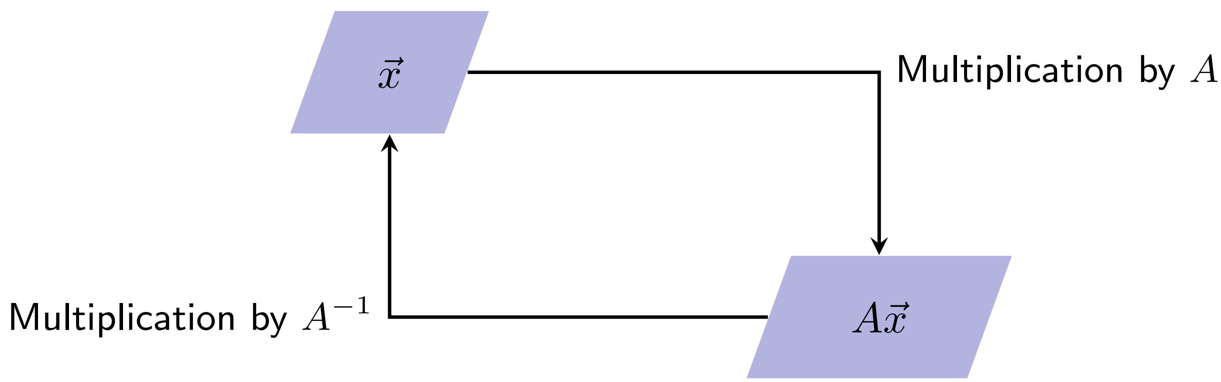
Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

a) *A is invertible.*
b) *A is row equivalent to I_n.*
c) *A has n pivotal columns. (All columns are pivotal.)*
d) *$A\vec{x} = \vec{0}$ has only the trivial solution.*
e) *The columns of A are linearly independent.*
f) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
g) *The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.* why? ($\because A \in \mathbb{R}^{n\times n}$)
h) *The columns of A span $\mathbb{R}^n$.*
i) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is onto.*
j) *There is a $n \times n$ matrix C so that $CA = I_n$. (A has a left inverse.)*
k) *There is a $n \times n$ matrix D so that $AD = I_n$. (A has a right inverse.)*
l) *A^T is invertible.*

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.



The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} =$$

The Invertible Matrix Theorem: Final Notes

- Items j and k of the invertible matrix theorem (IMT) lead us directly to the following theorem.

Theorem

If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

Example 1

Is this matrix invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

$$\xrightarrow[\substack{E_1:\ R_2 - 3R_1 \to R_2 \\ E_2:\ R_3 + 5R_1 \to R_3}]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow[\substack{E_3:\ R_2 + R_3 \to R_3 \\ \frac{1}{3}R_3 \to R_3}]{} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 + 2R_3 \to R_1 \\ R_2 - 4R_3 \to R_2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is invertible.

$$[\,A \mid I_n\,] \longrightarrow [\,I_n \mid A^{-1}\,]$$

$$\cdots \underbrace{\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{E_3} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}}_{E_2} \cdot \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E_1}} = A^{-1}$$

Example 2

Invertible = non-singular.

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

Not Invertible.

triangular matrix

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \underbrace{\begin{pmatrix} 1 & \boxed{?} & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{NP.}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Columns are linearly dependent

= A is Not Invertible.

$$\underset{\substack{\shortparallel \\ 0}}{x_1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \underset{\substack{\shortparallel \\ 0}}{x_2}\begin{bmatrix} ? \\ 1 \\ 0 \end{bmatrix} + \underset{\substack{\shortparallel \\ 0}}{x_3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Section 2.4 : Partitioned Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
- William Paul Thurston

Multiple perspectives of the same concept is a theme of this course; each perspective deepens our understanding. In this section we explore another way of representing matrices and their algebra that gives us another way of thinking about them.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Partitioned matrices (or block matrices)

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication.

What is a Partitioned Matrix?

(block)

Example
This matrix:

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

can also be written as:

$$\begin{bmatrix} \overset{A_{1,1}}{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{bmatrix}} & \overset{A_{1,2}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Another Example of a Partitioned Matrix

Example: The reduced echelon form of a matrix. We can use a partitioned matrix to

Identity

$$\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right] = \begin{bmatrix} I_4 & F \\ 0 & 0 \end{bmatrix}$$

0 0

This is useful when studying the **null space** of A, as we will see later in this course.

$= \{ \vec{x} : A\vec{x} = 0 \}$

Row Column Method

Recall that a row vector times a column vector (of the right dimensions) is a scalar. For example,

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} =$$

This is the **row column** matrix multiplication method from Section 2.1.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\mathrm{row}_i\, A \cdot \mathrm{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions).

Ex

$$\begin{array}{c} n \\ n \end{array}\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]\begin{array}{c} n \\ n \end{array} = \left[\begin{array}{c|c} AP+BR & AQ+BS \\ \hline CP+DR & C \cdot Q+DS \end{array}\right]$$

like 2x2 matrices product

Example of Row Column Method

$\frac{1}{a}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix} \frac{1}{c}$

Recall, using our formula for a 2×2 matrix, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \underbrace{\frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}}$.

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

$\frac{1}{AC} = (AC)^{-1}$

$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = ?$ $\quad \underbrace{(AC)^{-1}}_{n\times n} \underbrace{\begin{bmatrix} C & -B \\ 0 & A \end{bmatrix}}_{2n\times 2n} ?$

$\begin{bmatrix} (AC)^{-1}C & -(AC)^{-1}\cdot B \\ 0 & (AC)^{-1}\cdot A \end{bmatrix}$ ×

$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \cdot \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I_{2n} = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & I_n \end{array}\right]$

$P = A^{-1}$, $R = 0$, $S = C^{-1}$

$= \begin{bmatrix} AP + \cancel{BR} & AQ + BS \\ \cancel{CR} & CS \end{bmatrix}$

$AP = I$, $CR = 0$, $CS = I$

$AQ + BC^{-1} = 0$

$A^{-1}(AQ) = -A^{-1}(BC^{-1})$

$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} A^{-1} & Q \\ 0 & C^{-1} \end{bmatrix} = \left[\begin{array}{c|c} I & AQ + BC^{-1} \\ \hline 0 & I \end{array}\right]$ $\quad AQ + BC^{-1} = 0$

$\therefore Q = -A^{-1}BC^{-1}$

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} \cdot C \cdot C^{-1} & A^{-1}(-B) \cdot C^{-1} \\ 0 & A^{-1} \cdot A \cdot C^{-1} \end{bmatrix}$$

Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop.

Topics and Objectives

Topics
We will cover these topics in this section.

1. The LU factorization of a matrix
2. Using the LU factorization to solve a system
3. Why the LU factorization works

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Compute an LU factorization of a matrix.
2. Apply the LU factorization to solve systems of equations.
3. Determine whether a matrix has an LU factorization.

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using

$$\vec{x} = A^{-1}\vec{b}$$

- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

Triangular Matrices

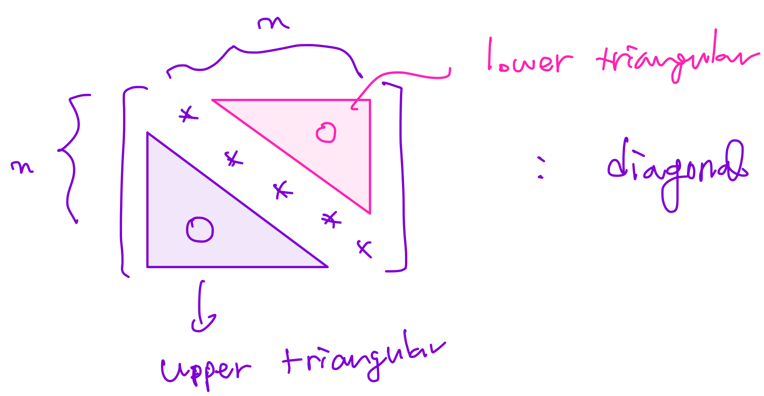
- A rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$.
  Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$.
  Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Ask: Can you name a matrix that is both upper and lower triangular?



: diagonal matrix.

The LU Factorization

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an **echelon** form of A.

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$\overbrace{}^{n=3} \qquad m = 3 \qquad A = LU = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}}_{\mathbb{R}^{m\times m},\ \mathbb{R}^{3\times 3}} \underbrace{\begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}}_{\mathbb{R}^{m\times n},\ \mathbb{R}^{3\times 2}}$$

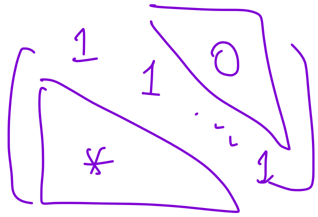
$A \xrightarrow[1]{E_1} \cdots \xrightarrow[p]{E_p} \qquad = U$

upper triangular

$R_1 - 5R_2 \to R_1$

- $\boxed{\text{Replacement}}$
- Swap
- Scalar multiple

$\boxed{E_p \cdots E_2 E_1} \cdot A = U$

$L =$ 

$$E_{p-1} \cdots E_1 A = E_p^{-1} \cdot U$$

$$E_{p-2} \cdots E_1 A = E_{p-1}^{-1} \cdot E_p^{-1} \cdot U$$

$$\vdots$$

$$A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_p^{-1}}_{= L \text{ lower triangular.}} U$$

$$= L \cdot U$$

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows. Then,

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm: construct $A = LU$, solve $A\vec{x} = LU\vec{x} = \vec{b}$ by:

1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

Example: Solve the linear system whose LU decomposition is given.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$A \in \mathbb{R}^{4\times 3}$

$y_1 + y_2 = 3$, $y_1 = 2$, $y_2 = 1$

$2y_2 + y_3 = 2$, $y_2 = 1$, $y_3 = 0$

$A x = \vec{b}$

$L\boxed{U\vec{x}} = \vec{b}$

① $L\boxed{\vec{y}} = \vec{b}$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

② $U\boxed{\vec{x}} = \vec{y} \rightarrow \ldots$

$A = m\Big\{\overbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}^{n} = L\cdot U = m\Big\{\underbrace{\begin{pmatrix} 1 & & 0 \\ & 1 & \\ * & \ddots & 1 \end{pmatrix}}_{m} \cdot \underbrace{\begin{pmatrix} * & & * \\ & * & \\ 0 & & * \end{pmatrix}}_{m}\Big\}n$

L: Inverse of Elementary matrices

U: An echelon form of A

$A\vec{x} = \vec{b} = L\cdot(U\vec{x}) = \vec{b}$

① $L\cdot\vec{y} = \vec{b}$ (Find y_1 first, y_2, ...)

② $U\cdot\vec{x} = \vec{y}$ (Find x_n first, x_{n-1}, ...)

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note that

- In MATH 1554, the only row replacement operation we can use is to *replace a row with a multiple of a row above it*.
- More advanced linear algebra courses address this limitation.

Example: Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix} \in \mathbb{R}^{3\times 4}$$

$$\xrightarrow[\substack{R_2 + 4R_1 \to R_2 \text{ ①} \\ R_3 - 2R_1 \to R_3 \text{ ②}}]{} \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -10 & 12 \end{pmatrix} = U.$$

$$A \xrightarrow{①} E_1\cdot A \xrightarrow{②} \underbrace{E_2E_1A = U}$$

$I \xrightarrow{①} E_1$ $\qquad$ $I \xrightarrow{②} E_2$

$$A = \underbrace{E_1^{-1}\cdot E_2^{-1}}_{= L} U$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_1^{-1} \cdot E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \qquad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad = L$$

Summary

- To solve $A\vec{x} = LU\vec{x} = \vec{b}$,
  1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
  2. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.
- To compute the LU decomposition:
  1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
  2. Place entries in L such that the same sequence of row operations reduces L to I.
- The textbook offers a different explanation of how to construct the LU decomposition that students may find helpful.
- Another explanation on how to calculate the LU decomposition that students may find helpful is available from MIT OpenCourseWare: www.youtube.com/watch?v=rhNKncraJMk

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question

Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

Subsets of $\mathbb{R}^n$

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

Subspaces in $\mathbb{R}^n$

= Subsets with algebraic structure

(vector addition + Scalar multiple)

Definition

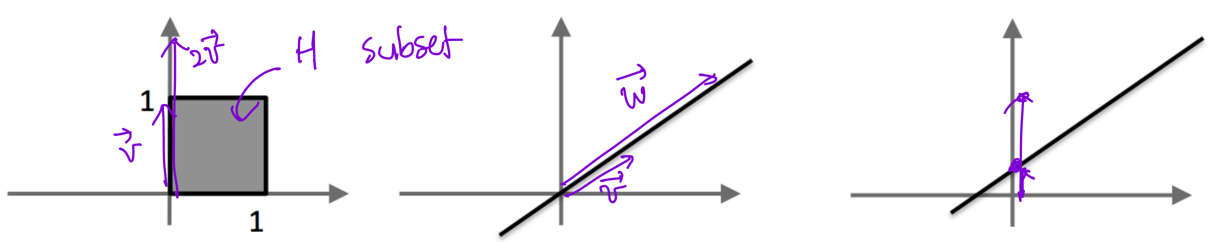
A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\vec{u} \in H$
2. $\vec{u} + \vec{v} \in H$

If $c = 0$. $c \cdot \vec{v} = \vec{0} \in H \Rightarrow$ Every subspace contains $\vec{0}$

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square

subspace ?

No.

$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in H$

$2\vec{v} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin H$

b) a line passing through the origin

subspace ?

Yes.

$\vec{0} \in H$

$c_1\vec{v} + c_2\vec{v}$

$= (c_1 + c_2)\vec{v} \in H$

c) a line that doesn't pass through the origin

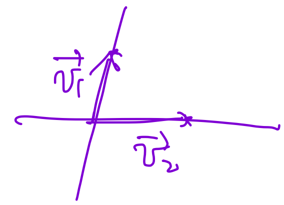
Subspace ?

No !

$\vec{0} \notin H$

Note

① $\mathbb{R}^n$ is a subspace? Yes



② $\{\vec{0}\}$ is a subspace.

③ $\{c \cdot \vec{v} : c \in \mathbb{R}\}$: subspace : a straight line.

$\text{Span}\{v_1, v_2\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 : c_1, c_2 \in \mathbb{R}\}$

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$, that $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is:

This is a **subspace**, spanned by $\vec{v}_1, \ldots, \vec{v}_p$.

Note: $\text{Span}\{v_1, \cdots, v_p\}$ is the smallest subspace containing $v_1, \cdots, v_p$

Definition

Given an $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

1. The **column space of** A, $\text{Col}\, A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \ldots, \vec{a}_n$. $= \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$
2. The **null space of** A, $\text{Null}\, A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

Column space of A $= \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$

$= \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ $\quad T(\vec{x}) = A\vec{x}$

$=$ Range of T.

Null space of A $= \{\vec{x} : A\vec{x} = 0\}$

$=$ Solution set of $A\vec{x} = \vec{0}$

Example

Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

$\underline{\vec{b}} \in \mathrm{Col}(A) = \mathrm{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

$= \{A\vec{x} : \vec{x} \in \mathbb{R}^3\}$

There exists $\vec{x}$ s.t. $A\vec{x} = \vec{b}$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \quad & \quad \end{array}\right]$$

pivot or Nonpivot.

Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \underline{\lambda \in \mathbb{R}}$$

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$$

$$\vec{v} \in \text{Null}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$$

$$A \cdot \vec{v} = ?$$

$$A \cdot \begin{bmatrix} -5\lambda \\ -3\lambda \\ 1\lambda \end{bmatrix} = \lambda \begin{pmatrix} -5+9-4 \\ 20-18-2 \\ 15-21+6 \end{pmatrix} = \vec{0}$$

Basis

$\{v_1, v_2, \cdots, v_p\} \subseteq H$ s.t. lin. indep. ; $\mathrm{Span}\{v_1, \cdots, v_p\} = H$.

a subset $\overbrace{\ \ }^{\text{of } \mathbb{R}^n}$ with $\vec{u}+\vec{v} \in H$; $c \cdot \vec{u} \in H$

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

The set $H = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0\}$ is a subspace.

a) H is a null space for what matrix A?

b) Construct a basis for H.

Ex $\mathbb{R}^2$ is a subspace

$\{\vec{e}_1, \vec{e}_2\} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \leftarrow$ lin. indep. & $\mathrm{Span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$

$\Rightarrow \{e_1, e_2\}$ is a basis.

Basis

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

$\mathrm{Null}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$

Example

$\mathrm{Null}(A)$ $''$

The set $H = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0 \right\}$ is a subspace.

$x_1 = -2x_2 - x_3 - 5x_4$

$= A \cdot \vec{x} = \begin{bmatrix} 1 & 2 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

a) H is a null space for what matrix A?

b) Construct a basis for H.

$\{v_1, \dots, v_p\} \subseteq H$ s.t. $\Big\langle$ $\mathrm{Span}\{v_1, \dots, v_p\} = H$; linearly indep.

How to describe solution set using finitely many vectors.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 - 5x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_4 \\ 0 \\ 0 \\ x_4 \end{bmatrix}$$

parametric vector form $\rightarrow$

$$= x_2 \cdot \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{=v_1} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{=v_2} + x_4 \underbrace{\begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{=v_3}$$

$H = \text{Span}\{v_1, v_2, v_3\}$ / v_1, v_2, v_3 : linearly indep.

$\Rightarrow$ $\{v_1, v_2, v_3\}$: a basis for H.

Example

$\{\vec{x} : A\vec{x} = \vec{0}\}$ $\quad$ Span{ columns of A }

$\quad$ " $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ "

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

① Null (A) :

$$\begin{cases} x_1 - 2x_2 = 0 \\ x_3 = 0 \end{cases} \qquad x_1 = 2x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of Null (A).

Note : # of basis = # of free variables

= # of non pivot columns.

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

② $\text{Col}(A) \overset{\text{WANT}}{=} \text{Span}\{\underbrace{v_1, \dots, v_p}_{\text{lin. indep.}}\}$

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \cancel{\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$$= \text{Span}\left\{ \underbrace{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}}_{\text{basis}} \right\}$$

Note: # of basis of Col(A) = # of pivot columns.

Additional Example

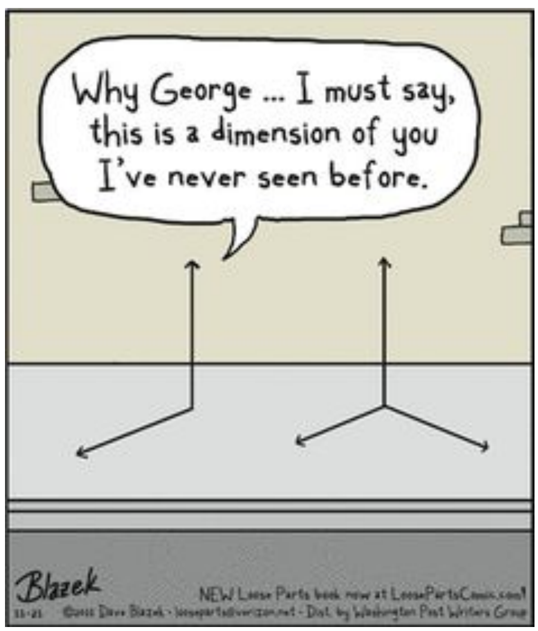
Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$.

1. Give an example of a vector that is in V.
2. Give an example of a vector that is not in V.
3. Is the zero vector in V?
4. Is V a subspace?

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics
We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives
For the topics covered in this section, students are expected to be able to do the following.

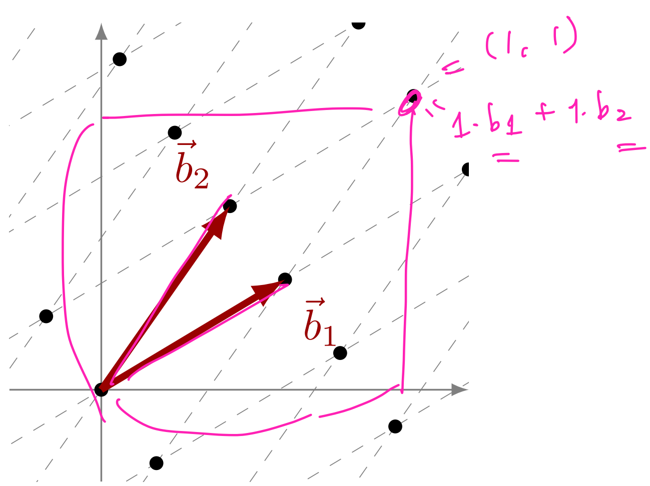
1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

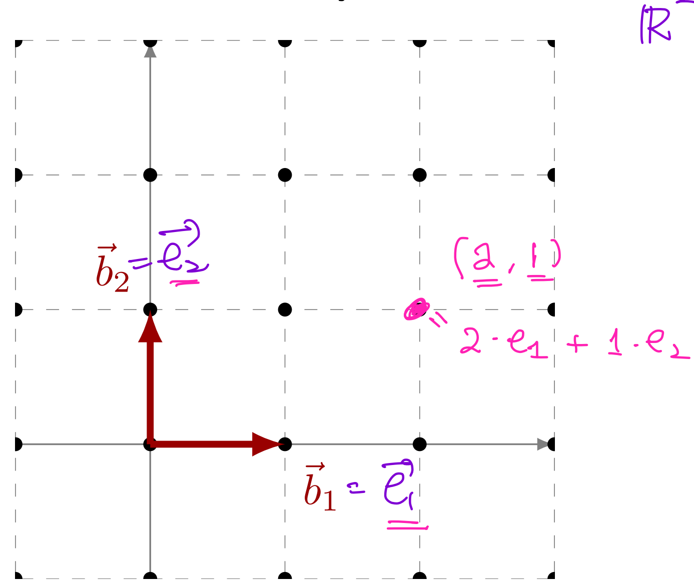
Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.

$\mathbb{R}^2$





Coordinates

$H = \operatorname{Span} \mathcal{B}$

$\vec{x} \in$

Definition

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \ldots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinate vector of $\vec{x}$**

Ex $\quad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= -1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Example 1

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

Q1: $\vec{x} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$?

Find x_1, x_2 such that

$$\underset{2}{x_1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \underset{3}{x_2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

last column is non pivot $\Rightarrow$ there is a solution.

$\Rightarrow \vec{x} \in \text{Span}\, \mathcal{B}$.

Q2: $\mathcal{B}$- coordinate : WANT TO FIND x_1, x_2

$$\begin{cases} x_1 + x_2 = 5 \\ x_2 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases} \Rightarrow [x]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Dimension

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples: basis ? $\{\vec{e}_1, \dots, \vec{e}_n\}$

1. $\dim \mathbb{R}^n =$ n

   $A = [1 \ 1 \ \cdots \ 1]$

2. $H = \{(x_1, \dots, x_n) \ : \ x_1 + \cdots + x_n = 0\}$ has dimension $= n-1$

   $= \text{Null}(A)$ $\quad = \#$ of free variables $= \#$ of nonpivots

3. $\dim(\text{Null}\, A)$ is the number of

   non-pivots

4. $\dim(\text{Col}\, A)$ is the number of

   pivots

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0 \right\} = \text{Null}\left([1 \ 1 \ 1 \ 1] \right)$$

$$[\underset{\text{pivot}}{1} \ \underbrace{1 \ 1 \ 1}_{\text{non pivot}}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} \\ \\ \end{bmatrix} + x_3 \begin{bmatrix} \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \end{bmatrix}$$

· Subspaces ex) $\mathbb{R}^n$, $\{\vec{0}\}$, Span$\{\vec{v}_1, \dots, \vec{v}_p\}$, Null space of A $= \{\vec{x} : A\vec{x} = 0\}$

Col(A) = Span{Columns of A}

· a basis of H : $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\} \subseteq H$ $\begin{cases} \text{Span}\{v_1, \dots, v_p\} = H \\ \text{linearly independent.} \end{cases}$

· # of vectors in a basis = dimension

Note : (i) dimension does not depend on a choice of basis.

(ii) dim H = 0 $\Leftrightarrow$ $H = \{\vec{0}\}$ ($H \neq \phi$)

Rank

rank(A) = dim (Col(A))

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \overset{\text{row operation}}{\sim \cdots \sim} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

① dim (Null (A)) = 2 = # of free variables = # of nonpivots.

Null(A) = $\{\vec{x} : A\vec{x} = \vec{0}\}$

x_3, x_5 : free variables

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} = \begin{bmatrix} - - - \\ - - - \\ x_3 \\ - - - \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} v_1 \end{bmatrix} + x_5 \begin{bmatrix} v_2 \end{bmatrix}$$

$\{v_1, v_2\}$: a basis of Null(A)

Rank

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{array}{ccccc} \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \end{array} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\vec{a_1}$ " $\vec{a_2}$ " $\vec{a_3}$ " $\vec{a_4}$ " $\vec{a_5}$

② $\text{rank}(A) = \dim(\text{Col}(A))$

Find a basis of Col(A)

$\text{Col}(A) = \text{span}\{\underline{\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}, \vec{a_5}}\}$ linearly indep?

Recall $\{v_1, v_2, \dots, v_p\}$ are linearly indep.

$\Leftrightarrow$ $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$ implies

$x_1 = x_2 = \cdots = x_p = 0$.

$\Leftrightarrow$ $[\vec{v_1} \cdots \vec{v_p}] \cdot \vec{x} = 0$ has the only trivial solution

$\Leftrightarrow$ $[v_1 \cdots v_p]$ has no free variable

(nonpivot columns

$A = \begin{bmatrix} a_1 & a_2 & \cancel{a_3} & a_4 & \cancel{a_5} \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 5 & & -4 \\ 0 & -3 & & 5 \\ 0 & 0 & & 4 \\ 0 & 0 & & 0 \end{bmatrix}$

$\Rightarrow \{a_1, a_2, a_4\}$: linearly indep. pivot

Rank

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$Col(A) = Span\{a_1, a_2, a_3, a_4, a_5\}$

$\overset{?}{=}$?!! $Span\{a_1, a_2, a_4\}$ linearly indep.

$\Leftarrow$ Need: $a_3 \in Span\{a_1, a_2, a_4\}$
a_5

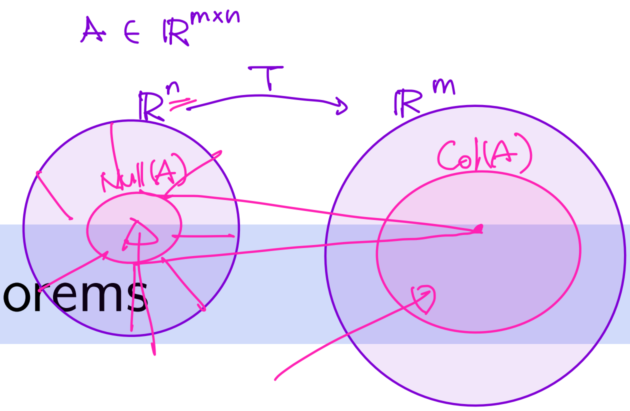
$\begin{bmatrix} a_1 & a_2 & a_4 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} a_3 \end{bmatrix}$

consistent ? Yes

if the last column is nonpivot.

a basis of $Col(A) = \{$ pivot columns $\}$

$rank(A) = \dim(Col(A)) = \#$ of pivots.



Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

If a matrix A has n columns, then $\mathrm{Rank}\, A + \dim(\mathrm{Nul}\, A) = n$.

Theorem (Basis Theorem)

Any two bases for a subspace have the same dimension.

Theorem (Invertibility Theorem)

Let A be a $n \times n$ matrix. These conditions are equivalent.

IMT

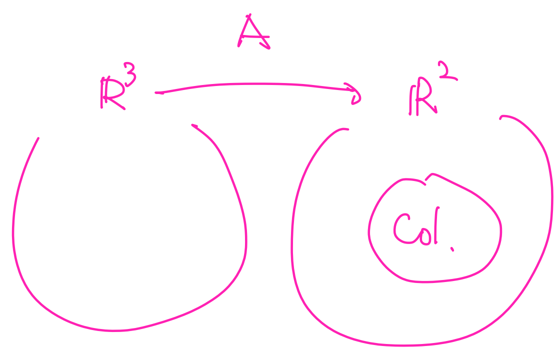
1. *A is invertible.*
2. *The columns of A are a basis for $\mathbb{R}^n$.*
3. $\mathrm{Col}\, A = \mathbb{R}^n$.
4. $\mathrm{rank}\, A = \dim(\mathrm{Col}\, A) = n$.
5. $\mathrm{Null}\, A = \{0\}$.

Examples

If possible give an example of a 2×3 matrix A, that is in RREF and has the given properties.

a) $\text{rank}(A) = 3$

N.P.



b) $\text{rank}(A) = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

c) $\dim(\text{Null}(A)) = 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

d) $\text{Null}A = \{0\}$

NP

$A\vec{x} = \vec{0}$ has only trivial solution.

$\boxed{T(\vec{x}) = A\vec{x} \text{ is } 1-1.}$