Practice for Exam 2: Solution

MATH 3215, Summer 2023

1. Let X and Y be two discrete random variables with joint pmf $p_{X,Y}(1,1) = p_{X,Y}(1,2) = \frac{1}{3}$, $p_{X,Y}(2,1) = p_{X,Y}(2,2) = \frac{1}{6}$. Find $\mathrm{Cov}(X,Y)$ and ρ. Are they independent?
ANS: $\mathrm{Cov}(X,Y) = 0 = \rho$ and they are independent.

2. Let X and Y be two discrete random variables with joint pmf $p_{X,Y}(x,y) = \frac{x+y}{27}$, where $x = 2,3$ and $y = 1,2,3$. Find the marginal pmfs of X. Find the conditional expectation of X given $Y = 2$.
ANS: $p_X(x) = \frac{x+2}{9}$, $p_Y(y) = \frac{2y+5}{27}$, and $\mathbb{E}[X|Y=2] = \frac{23}{9}$.

3. Let X and Y be two random variables with joint pdf $f_{X,Y}(x,y) = 2e^{-x-y}$ for $0 \le x \le y < \infty$. Find the marginal pdfs of X and Y. Are they independent?
ANS: $f_X(x) = 2e^{-2x}$ and $f_Y(y) = 2e^{-y}(1-e^{-y})$. They are dependent.

4. (There were typos in the problem. Corrected.) Let X and Y be two random variables with joint pdf $f_{X,Y}(x,y) = \frac{1}{2}$ for $0 < x+y < 2$, $x > 0$, $y > 0$. Find the conditional expectation $\mathbb{E}[X|Y]$.
ANS: $\mathbb{E}[X|Y] = 1 - \frac{Y}{2}$.

5. Let X be a uniform random variable on $(-1,3)$ and $Y = X^2$. Find the pdf of Y.
ANS:
$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 \le y < 1, \\ \frac{1}{8\sqrt{y}}, & 1 \le y < 9, \\ 0, & \text{otherwise.} \end{cases}$$

6. Let X be a random variable with pdf given by
$$f_X(x) = x^2 + \frac{10x^4}{3}$$
for $0 < x < 1$, and otherwise $f_X(x) = 0$. Find the cdf and the pdf of $Y = \log X$.
ANS:
$$F_Y(y) = \begin{cases} \frac{1}{3}(e^{3y} + 2e^{5y}), & y < 0, \\ 1, & y \ge 0, \end{cases}$$
and $f_Y(y) = e^{3y} + \frac{10}{3}e^{5y}$ for $y < 0$ and otherwise 0.

7. Let X be an exponential random variable with parameter 1, i.e., its pdf is given by $f_X(x) = e^{-x}$, $x > 0$. Find the pdf of $Y = X^4$. Are X and $Y^{\frac{1}{2}}$ negatively correlated?
ANS: $f_Y(y) = \frac{1}{4}y^{-\frac{3}{4}}e^{-t^{\frac{1}{4}}}$ for $y \ge 0$ and otherwise 0. $\mathrm{Cov}(X, Y^{1/2}) = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2] = 6 - 2 = 4 > 0$.

8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}$$

for $0 \leq x < \infty$. Find the expected lifetime of such a tube.
ANS: $\mathbb{E}[X] = 2$.

9. Let (X,Y) be a bivariate normal random vector. Both X and Y have mean 1 and variance 4, while the correlation coefficient of X and Y is $\rho = \frac{1}{4}$. Find $\text{Var}(-2X+Y)$. Assume now that (X,Y) is a bivariate normal vector, with X and Y having mean 9 and variance 9, but that the correlation coefficient has changed and is now given by $\rho = 0$. Find $\mathbb{P}(3 \leq X \leq 15, Y \leq 9)$ and using the tables, find an approximate value for it.
ANS: $\text{Var}(-2X+Y) = 4\,\text{Var}(X) + \text{Var}(Y) - 4\,\text{Cov}(X,Y) = 16$. $\mathbb{P}(3 \leq X \leq 15, Y \leq 9) = \mathbb{P}(Z \leq 2) - \frac{1}{2} \approx 0.4772$.

10. The life of a certain type of automobile tire is normally distributed with mean 34,000 miles and standard deviation 4000 miles.

    (a) What is the probability that such a tire lasts more than 40,000 miles?
    (b) What is the probability that it lasts between 30,000 and 35,000 miles?
    (c) Given that it has survived 30,000 miles, what is the conditional probability that the tire survives another 10,000 miles?

    ANS: (a) $\mathbb{P}(Z \geq 1.5) \approx 0.0668$
    (b) $\mathbb{P}(-1 \leq Z \leq 0.25) \approx 0.44$
    (c) $\mathbb{P}(Z \geq 1.5)/\mathbb{P}(Z \geq -1) \approx 0.0794$

11. Let X be a random variable with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find the density of $1/X$.
ANS: $f_X = f_{1/X}$.