Chapter 5. Distributions of Functions of Random Variables

Math 3215 Spring 2024

Georgia Institute of Technology

Section 1.
Functions of One Random Variable

Functions of One Random Variable

Let X be a random variable.

Define $Y = u(X)$ for some function u.

We discuss how to find the distribution of Y from that of X.

Example

Let X have a discrete uniform distribution on the integers from -2 to 5.

Find the distribution of $Y = X^2$.

CDF Technique

Example

Let X have a gamma distribution with PDF

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}}.$$

Find the distribution of $Y = e^X$.

CDF Technique

Theorem

Let X be a random variable with CDF F.

Suppose F is strictly increasing, $F(a) = 0$, $F(b) = 1$.

Let $Y \sim U(0, 1)$.

Then, $X = F^{-1}(Y)$.

Change of Variables

Example

Let X have the PDF $f(x) = 3(1-x)^2$ for $0 < x < 1$.

Find the distribution of $Y = (1-X)^3$.

Exercise

Let X have the PDF $f(x) = 4x^3$ for $0 < x < 1$.

Find the PDF of $Y = X^2$.

Section 2.
Transformations of Two Random Variables

Transformations of Two Random Variables

If X_1 and X_2 are two continuous-type random variables with joint PDF $f(x_1, x_2)$.

Let $Y_1 = u_1(X_1, X_2)$, $Y_2 = u_2(X_1, X_2)$.

If $X_1 = v_1(Y_1, Y_2)$, $X_2 = v_2(Y_1, Y_2)$, then the joint PDF of Y_1 and Y_2 is

$$f_{Y_1,Y_2} = |J| f_{X_1,X_2}(v_1(y_1, y_2), v_2(y_1, y_2))$$

where J is the Jacobian given by

$$J := \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}.$$

Transformations of Two Random Variables

Example

Let X_1 and X_2 have the joint PDF

$$f(x_1, x_2) = 2, \qquad 0 < x_1 < x_2 < 1.$$

Find the joint PDF of $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.

Exercise

Let X_1 and X_2 be independent random variables, each with PDF

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Find the joint pdf of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Section 3.
Several Independent Random Variables

Independent random variables

Recall that X_1 and X_2 are independent if

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A)\mathbb{P}(X_2 \in B)$$

for all A, B.

In particular, if X_1 and X_2 have PDFs, then $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$.

Independent random variables

Definition

In general, we say $X_1, X_2, \cdots, X_n$ are independent if $\{X_1 \in A_1\}, \{X_2 \in A_2\}, \cdots, \{X_n \in A_n\}$ are mutually independent, for any choice of $A_1, A_2, \cdots, A_n$.

In particular, if $X_1, X_2, \cdots, X_n$ has PDFs, then the joint PDF is the product.

If $X_1, X_2, \cdots, X_n$ are independent and have the same distribution,

we say they are i.i.d. (independent and identically distributed) or a random sample of size n from that common distribution.

Independent random variables

Example

Let X_1, X_2, X_3 be a random sample from a distribution with PDF

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Find $\mathbb{P}(0 < X_1 < 1, 2 < X_2 < 4, 3 < X_3 < 7)$.

Expectation and Variance

Theorem

Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables. Then,

$$\mathbb{E}[X_1 + X_2 + \cdots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n].$$

If they are independent, then

$$\mathbb{E}[X_1 X_2 \cdots X_n] = \mathbb{E}[X_1]\mathbb{E}[X_2] \cdots \mathbb{E}[X_n]$$

and

$$\mathrm{Var}[X_1 + X_2 + \cdots + X_n] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n].$$

Exercise

Let X_1, X_2, X_3 be i.i.d. Geometric with $p = \frac{3}{4}$.

Let Y be the minimum of X_1, X_2, X_3.

Find $\mathbb{P}(Y > 4)$.

Section 4.
The Moment-Generating Function Technique

The Moment-Generating Function

Theorem

If $X_1, X_2, \cdots, X_n$ are independent and have the MGFs $M_{X_i}(t)$, then the MGF of $Y = a_1X_1 + \cdots a_nX_n$ is $M_Y(t) = M_{X_1}(a_1t) \cdots M_{X_n}(a_nt)$.

Theorem

If $X_1, X_2, \cdots, X_n$ are i.i.d., then the MGF of $Y = X_1 + \cdots + X_n$ is $M_Y(t) = M_X(t)^n$.
If $\overline{X} = \frac{X_1 + \cdots + X_n}{n}$, then the MGF is $M_{\overline{X}}(t) = M_X(\frac{t}{n})^n$.

The Moment-Generating Function

Example

Let $X_1, X_2, \cdots, X_n$ be i.i.d. Bernoulli with p.

Let $Y = X_1 + \cdots + X_n$.

Find the MGF of Y.

The Moment-Generating Function

Example

Let $X_1, X_2, \cdots, X_n$ be i.i.d. exponential with θ.

Let $Y = X_1 + \cdots + X_n$.

Find the MGF of Y.

Exercise

Let X_1, X_2, X_3 be independent Poisson with means $2, 1, 4$.

Find the MGF of $Y = X_1 + X_2 + X_3$.

Section 6.
The Central Limit Theorem

The Central Limit Theorem

Let $X_1, X_2, \cdots, X_n$ be i.i.d. with common distribution X.

Let $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

Let $\overline{X} = \frac{X_1 + \cdots + X_n}{n}$, then

$\mathbb{E}[\overline{X}] =$

$\text{Var}(\overline{X}) =$

Let $W = \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$, then

$\mathbb{E}[W] =$

$\text{Var}(W) =$

The Central Limit Theorem

Theorem

If μ and σ^2 are finite, then the distribution of $W = \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$ converges to that of the standard normal distribution $N(0,1)$ as $n \to \infty$.

The convergence is in the following sense: If n is large, for the standard normal Z,

$$\mathbb{P}(W \leq x) \approx \mathbb{P}(Z \leq x) =: \Phi(x) = \int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{|y|^2}{2}}\, dy.$$

The Central Limit Theorem

Example

Let $\overline{X}$ be the mean of a random sample of $n = 25$ currents (in milliamperes) in a strip of wire in which each measurement has a mean of 15 and a variance of 4.

Find the approximate probability $\mathbb{P}(14.4 < \overline{X} < 15.6)$.

The Central Limit Theorem

Example

Let $\overline{X}$ denote the mean of a random sample of size 25 from the distribution whose PDF is $f(x) = \frac{x^3}{4}$, $0 < x < 2$.

Find the approximate probability $\mathbb{P}(1.5 \leq \overline{X} \leq 1.65)$.

Exercise

Let X equal the maximal oxygen intake of a human on a treadmill, where the measurements are in milliliters of oxygen per minute per kilogram of weight.

Assume that, for a particular population, the mean of X is $\mu = 54.030$ and the standard deviation is $\sigma = 5.8$.

Let $\overline{X}$ be the sample mean of a random sample of size $n = 47$.

Find $P(52.761 \leq \overline{X} \leq 54.453)$, approximately.

Section 8.
Chebyshev's Inequality and Convergence in Probability

Chebyshev's Inequality

Theorem

If the random variable X has a mean μ and variance σ^2, then for every $k \geq 1$,

$$\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

In particular $\varepsilon = k\sigma$, then

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Chebyshev's Inequality

Example

Suppose X has a mean of 25 and a variance of 16.

Find the lower bound of $\mathbb{P}(17 < X < 33)$.

The Law of Large Numbers

Definition

We say a sequence of random variables X_n converges to a random variable X in probability if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

The Law of Large Numbers

Theorem

Let $X_1, X_2, \cdots, X_n$ be i.i.d. with common distribution X.

Let $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

Then, $\overline{X}$ converges to μ in probability.

Exercise

If X is a random variable with mean 3 and variance 16, use Chebyshev's inequality to find

1. A lower bound for $\mathbb{P}(23 < X < 43)$.
2. An upper bound for $\mathbb{P}(|X - 31| \geq 14)$.