Math 3215: Intro to Probability and Statistics

Final Exam, Summer 2023

Date: July 31, 2023

NAME:______________________________ ID:______________________________

READ THE FOLLOWING INFORMATION.

- This is a 170-minute.
- This exam contains 16 pages (including this cover page) and 12 questions. Total of points is 100.
- Books, notes, and other aids are not allowed.
- Show all steps to earn full credit.
- Do not unstaple pages. Loose pages will be ignored.

Name	**PMF**	**Mean**	**Variance**
Ber(p)	$\mathbb{P}(X=1)=p, \mathbb{P}(X=0)=1-p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$ for $x=0,1,\ldots,n$	np	$np(1-p)$
Geom(p)	$p(1-p)^{x-1}$ for $x=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\binom{x-1}{r-1}p^r q^{x-r}$ for $x=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}$ for $x=0,1,\ldots$	λ	λ
Uniform(a,b)	$\frac{1}{b-a}$ for $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ for $x\in(-\infty,\infty)$	μ	σ^2
Exp(λ)	$\lambda e^{-\lambda x}$ for $x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(a,λ)	$\frac{\lambda^a x^{a-1}e^{-\lambda x}}{\Gamma(a)}$ for $x>0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$

Geometric series	$\sum_{k=0}^{N} ap^k = \frac{a(1-p^{N+1})}{1-p}$ for $p\in(0,1)$
Power series for e^x	$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

1. Suppose A, B, and C are events with

$$\begin{aligned} \mathbb{P}(A) = 0.58, \quad & \mathbb{P}(B) = 0.55, \quad \mathbb{P}(C) = 0.4, \\ \mathbb{P}(A \cap B) = 0.28, \quad & \mathbb{P}(A \cap C) = 0.23, \quad \mathbb{P}(B \cap C) = 0.23, \\ & \mathbb{P}(A \cap B \cap C) = 0.13. \end{aligned}$$

   (a) (4 points) Find $\mathbb{P}(C | A \cap B)$.

   (b) (4 points) Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

2. Suppose you roll two 4 faced dice, with faces labeled 1,2,3,4, and each equally likely to appear on top. Let X_1 be the number from the first die and X_2 the number from the second die. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\}$.

   (a) (4 points) Find the PMF of Y_2.

   (b) (4 points) Find $\mathbb{E}[Y_1 + Y_2]$.

3. Suppose X is a random variable taking values in $S = \{0, 1, 2, 3, \dots\}$ with PMF $f_X(k) = \dfrac{C \cdot 2^k}{k!}$.

(a) (4 points) Find the constant C.

(b) (4 points) Find $\mathbb{E}[e^{3X}]$.

4. Consider an urn containing 10 balls, of which 5 are black and 5 are white. Suppose two balls are drawn at random without replacement. Let A be the event that the first ball is black, and B the event that the second ball is white.

   (a) (4 points) Find $\mathbb{P}(B)$.

   (b) (4 points) Find $\mathbb{P}(A|B)$.

5. Let X and Y be two random variables with joint PDF $f_{X,Y}(x,y) = 2e^{-x-y}$ for $0 \leq x \leq y < \infty$ and otherwise 0.
   (a) (4 points) Find the marginal PDF of X.

   (b) (4 points) Find the conditional expectation $\mathbb{E}[Y|X = x]$ for $x > 0$.

6. Let X and Y be two random variables with joint PDF $f_{X,Y}(x,y) = 2$ for $0 < x+y < 1$, $x > 0$, $y > 0$, and otherwise 0.

   (a) (5 points) Find the covariance $\mathrm{Cov}(X,Y)$.

   (b) (5 points) Compute $\mathbb{P}(X \leq \frac{1}{2})$, $\mathbb{P}(Y \leq \frac{1}{2})$, and $\mathbb{P}(X \leq \frac{1}{2}, Y \leq \frac{1}{2})$.

7. Let (X,Y) be a bivariate normal random vector.

   (a) (5 points) Suppose that both X and Y have mean 2 and variance 3, while the correlation coefficient of X and Y is $\rho = -\frac{1}{6}$. Find $\mathrm{Var}(X-Y)$.

   (b) (5 points) Assume now that X and Y have mean 7 and variance 4, but that the correlation coefficient has changed and is now given by $\rho = 0$. Write $\mathbb{P}(4 \le X \le 9, Y \le 8)$ in terms of $\underline{\Phi(z) = \mathbb{P}(Z \le z) \text{ for } z \ge 0}$ where Z is the standard normal random variable.

8. Let X be a uniform random variable on $(-1,1)$ and $Y = X^3$.

   (a) (4 points) Find the pdf of Y.

   (b) (4 points) Compute $\mathrm{Cov}(X,Y)$.

9. A fair die will be rolled 720 times independently.

   (a) (5 points) What is the probability that among the 180 rolls the number 6 will appear between 27 and 32 times inclusively? That is, what is $\mathbb{P}(27 \leq X \leq 32)$? Write down the probability without using the tables and approximations.

   (b) (5 points) Using a normal approximation, **with half-unit correction**, write down an expression for the probability that among the 180 rolls the number 6 will appear between 27 and 32 times inclusively. Use the corresponding tables to find an approximate value for this probability.

10. (6 points) Let $\overline{X}$ be the mean of a random sample of size $n = 25$ from a distribution with mean $\mu = 16$ and variance $\sigma^2 = 63$. Use Chebvshev's inequality to find a lower bound for $\mathbb{P}(14 < \overline{X} < 18)$.

11. Let $W_1 < W_2 < \cdots < W_6$ be the order statistics of n independent observations from a $U(0,1)$ distribution.

(a) (5 points) Find the PDFs of W_1 and W_6.

(b) (5 points) Find $\mathbb{E}[W_1]$ and $\mathbb{E}[W_6]$.

12. (6 points) A random sample of size 100 from the normal distribution $N(\mu, 25)$ yielded $\overline{X} = 35.6$. Find a two-sided 95% confidence interval for μ.

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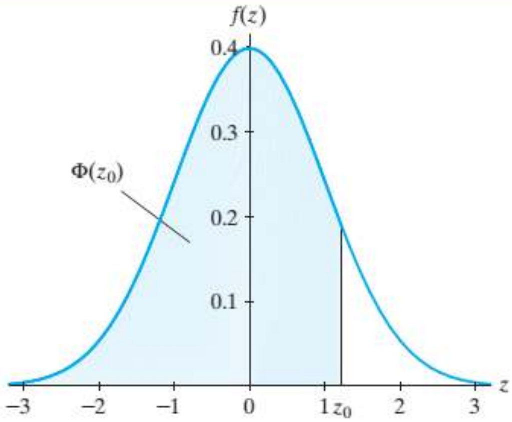
Table Va The Standard Normal Distribution Function



$$P(Z \le z) = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw$$

$$\Phi(-z) = 1 - \Phi(z)$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

α	0.400	0.300	0.200	0.100	0.050	0.025	0.020	0.010	0.005	0.001
z_α	0.253	0.524	0.842	1.282	1.645	1.960	2.054	2.326	2.576	3.090
$z_{\alpha/2}$	0.842	1.036	1.282	1.645	1.960	2.240	2.326	2.576	2.807	3.291

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