Homework 11 Solution

Math 461: Probability Theory, Spring 2022
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Due date: will not be graded

1. Suppose that the expected number of accidents per week at an industrial plant is 5. Suppose also that the numbers of workers injured in each accident are independent random variables with a common mean of 2.5. If the number of workers injured in each accident is independent of the number of accidents that occur, compute the expected number of workers injured in a week.

Solution: Let X_i be the number of workers injured in i-th accedent, and N be the number of accidents. We have $\mathbb{E}[X_i] = 2.5$ and $\mathbb{E}[N] = 5$. For each $n \geqslant 0$, we get

$$\mathbb{E}[\sum_{i=1}^{N} X_i | N = n] = n\,\mathbb{E}[X_1].$$

Thus,

$$\mathbb{E}[\sum_{i=1}^{N} X_i] = \mathbb{E}[\mathbb{E}[\sum_{i=1}^{N} X_i|N]] = \mathbb{E}[N\,\mathbb{E}[X_1]] = \mathbb{E}[N]\,\mathbb{E}[X_1] = 12.5.$$

2. The moment generating function for X is given by $M_X(t) = \exp(2e^t - 2)$ and that of Y by $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$. If X, Y are independent what are (a) $\mathbb{P}(X + Y = 2)$; (b) $\mathbb{P}(XY = 0)$ and (c) $\mathbb{E}(XY)$.

Solution: Since X is a random variable with moment generating function $M_X(t) = \exp\{2e^t - 2\} = \exp\{2(e^t - 1)\}$, X is Poisson with parameter $\lambda = 2$. Since Y is a random variable with moment generating function $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, Y is binomial with parameters $(10, \frac{3}{4})$.

(a)
$$\begin{aligned}\mathbb{P}\left(X+Y=2\right) &= \mathbb{P}\left(X=0\right)\mathbb{P}\left(Y=2\right) + \mathbb{P}\left(X=1\right)\mathbb{P}\left(Y=1\right) + \mathbb{P}\left(X=2\right)\mathbb{P}\left(Y=0\right) \\ &= e^{-2}\cdot\binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 2e^{-2}\cdot 10\frac{3}{4}\left(\frac{1}{4}\right)^9 + 2e^{-2}\cdot\left(\frac{1}{4}\right)^{10} \\ &= e^{-2}\left(\frac{1}{4}\right)^{10}(405+60+2) = \frac{467}{4^{10}e^2}.\end{aligned}$$

(b)
$$\begin{aligned}\mathbb{P}\left(XY=0\right) &= \mathbb{P}\left(X=0\right) + \mathbb{P}\left(Y=0\right) - \mathbb{P}\left(X=0\right)\mathbb{P}\left(Y=0\right) \\ &= e^{-2} + \frac{1}{4^{10}} - e^{-2}\frac{1}{4^{10}} = \frac{4^{10}+e^2-1}{4^{10}e^2}.\end{aligned}$$

(c)
$$\begin{aligned}\mathbb{E}(XY) &= \mathbb{E}(X)\cdot\mathbb{E}(Y) \quad \text{by independence} \\ &= 2\cdot 7.5 = 15.\end{aligned}$$

3. The joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi}}e^{-y}e^{-(x-y)^2/2} & \text{if } 0 < y < \infty, -\infty < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the joint moment generating function of X and Y.
(b) Compute the individual moment generating functions.

Solution: The joint moment generating function, $\mathbb{E}(e^{tX+sY})$ can be obtained either by using

$$\mathbb{E}(e^{tX+sY}) = \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{tx+sy} f(x,y)dydx$$

or by noting that Y is exponential with rate 1 and, given Y, X is normal with mean Y and variance 1. Hence, using this we obtain

$$\begin{aligned}\mathbb{E}(e^{tX+sY}) = \mathbb{E}(\mathbb{E}(e^{tX+sY} \mid Y)) &= \mathbb{E}(e^{sY}\,\mathbb{E}(e^{tX} \mid Y)) \\ &= \mathbb{E}(e^{sY} e^{tY+t^2/2}) = e^{t^2/2}\,\mathbb{E}(e^{(s+t)Y}) = e^{t^2/2}(1-s-t)^{-1}, \text{ when } s+t<1.\end{aligned}$$

Setting $s = 0$ we get

$$M_X(t) = \mathbb{E}(e^{tX}) = e^{t^2/2}(1-t)^{-1}, t < 1$$

and setting $t = 0$ we get

$$M_Y(s) = \mathbb{E}(e^{sY}) = (1-s)^{-1}, s < 1.$$

4. From past experience, a professor knows that the test score of a student taking her final examination is a random variable with mean 75.
(a) Give an upper bound for the probability that a student's test score will exceed 85. Suppose, in addition, that the professor knows that the variance of a student's test score is equal to 25.
(b) What can be said about the probability that a student will score between 65 and 85?
(c) How many students would have to take the examination to ensure, with probability at least .9, that the class average would be within 5 of 75? Do not use the central limit theorem.

Solution: (a) $P(X \geqslant 85) \leqslant \frac{75}{85} = \frac{15}{17}$.
(b) $P(65 \leqslant X \leqslant 85) = 1 - P(|X-75| > 10) \geqslant 1 - \frac{25}{100} = \frac{3}{4}$.
(c) Since

$$P\left(\left|\sum_{i=1}^{n} \frac{X_i}{n} - 75\right| > 5\right) \leqslant \frac{25}{25n},$$

we need $n = 10$.

5. Let $X_1, X_2, \ldots, X_{20}$ be independent Poisson random variables with mean 1.
(a) Use the Markov inequality to obtain a bound on

$$\mathbb{P}\left(\sum_{i=1}^{20} X_i > 25\right).$$

(b) Use the central limit theorem to approximate

$$\mathbb{P}\left(\sum_{i=1}^{20} X_i > 25\right).$$

Solution: (a) $P(\sum_{i=1}^{20} X_i > 25) \leqslant \frac{20}{25} = 0.8$.
(b) $P(\sum_{i=1}^{20} X_i > 25) = P(\sum_{i=1}^{20} X_i > 25.5) \approx P(Z > \frac{25.5-20}{\sqrt{20}}) = P(Z > 1.2298) \approx 0.1094$.

6. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$, approximate the probability that the resultant sum differs from the exact sum by more than 3.

Solution: Let X_i be the i-th roundoff error, then $E(\sum_{i=1}^{50} X_i) = 0$ and $\text{Var}(\sum_{i=1}^{50} X_i) = \frac{50}{12}$. Hence by the central limit theorem

$$P\left(|\sum_{i=1}^{50} X_i| > 3\right) \approx P(|Z| > \frac{3}{\sqrt{12/50}}) = 2P(Z > 1.47) = .1416.$$

7. A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.

Solution: If we let X_i be the lifetime of the i-th light bulb then the desired probability is

$$P\left(\sum_{i=1}^{100} X_i > 525\right).$$

By Central Limit theorem the desired probability is equal to, for $Z \sim$N(0,1),

$$P\left(Z > \frac{525 - 500}{\sqrt{2500}}\right) = P(Z > .5) = .3085.$$