In-Class Final Exam Review Set A, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

$Ax = b$

true	false	
●	○	If a linear system has more unknowns than equations, then the system has either no solutions or infinitely many solutions.
○	●	A $n \times n$ matrix A and its echelon form E will always have the same eigenvalues.
●	○	$x^2 - 2xy + 4y^2 \geq 0$ for all real values of x and y.
●	○	If matrix A has linearly dependent columns, then $\dim((\text{Row}A)^\perp) > 0$.
●	○	If λ is an eigenvalue of A, then $\dim\,(\text{Null}(A - \lambda I)) > 0$.
●	○	If A has QR decomposition $A = QR$, then $\text{Col}A = \text{Col}Q$.
●	○	If A has LU decomposition $A = LU$, then $\text{rank}(A) = \text{rank}(U)$.
●	○	If A has LU decomposition $A = LU$, then $\dim(\text{Null}\,A) = \dim(\text{Null}\,U)$).

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Null(A)

geom.

eigenspace

2. Give an example of the following.

   i) A 4×3 lower triangular matrix, A. such that $\text{Col}(A)^\perp$ is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}$. $\quad A = \begin{pmatrix} \quad \end{pmatrix}$

   $\underline{\text{Col}(A)}$ basis.

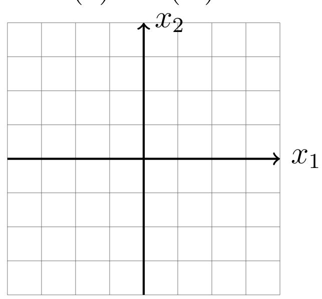
   $= \{ \vec{x} : \vec{x} \cdot \vec{v} = 0 \}$

   $x_1 + 2x_2 + 3x_3 + x_4 = 0$.

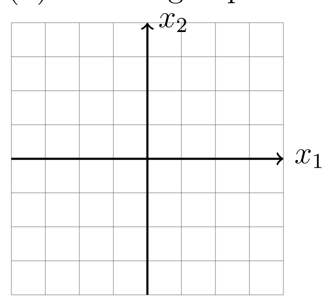
   ii) A 3×4 matrix A, that is in RREF, and satisfies $\dim\left((\text{Row}\,A)^\perp\right) = 2$ and $\dim\left((\text{Col}\,A)^\perp\right) = 2$. $A = \begin{pmatrix} \quad \end{pmatrix}$

3. (3 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. On the grid below, sketch a) Col(A), and b) the eigenspace corresponding to eigenvalue $\lambda = 5$.

(a) Col(A)



(b) $\lambda = 5$ eigenspace



$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$Q(\vec{x}) = 0x^2 - 2xy + 4y^2 = \vec{x}^T \cdot A \cdot \vec{x} \geq 0 \quad \forall \vec{x}$$

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

$\Updownarrow$

Q is positive s.d

$\Updownarrow$

$$\lambda^2 - (1+4)\lambda + (1 \cdot 4 - (-1)^2) = 0 \qquad \lambda \geq 0$$

$$\lambda^2 - 5\lambda + 3 = 0 = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$\lambda = \frac{5 \pm \sqrt{\cdots}}{2} \qquad = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \cdot \lambda_2$$

$$\lambda_1 + \lambda_2 = 5$$

$$\lambda_1 \cdot \lambda_2 = 3$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

$$x^2 - 2xy + 4y^2 = (x^2 - 2xy + y^2) + 3y^2$$

$$= \underbrace{(x-y)^2}_{\geq 0} + \underbrace{3y^2}_{\geq 0} \geq 0.$$

T/F If A then B

True :

False : Counter example

satisfies A

not B

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}$, basis for $(\text{Span}\{v\})^{\perp}$

$$A = [x_1 \; . \; . \; . \; . \; , \; x_n]$$

$$\downarrow \quad \text{G - S}$$

$$Q = [u_1 \; - \; - \; , \; - \; u_n] \quad \text{orthonormal}$$

$$R = \begin{bmatrix} x_1 \cdot u_1 & u_2 \cdot x_1 & \cdots \\ & u_2 \cdot x_2 & \\ 0 & & \end{bmatrix}$$

$$\text{Span}\{x_1\} = \text{Span}\{u_1\}$$

$$\text{Span}\{x_1, x_2\} = \text{Span}\{u_1, u_2\}$$

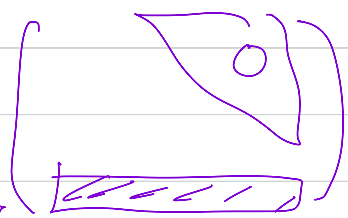
$$\vdots$$

$$\text{Col}(A) = \text{Span}\{x_1, \cdots, x_n\} = \text{Span}\{u_1, \cdots, u_n\} = \text{Col}(Q).$$

$$\Rightarrow \text{Col}(A) = \{ \vec{x} : \underline{x_1 + 2x_2 + 3x_3 + x_4 = 0} \} \quad ?$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

$$x_4 = -x_1 - 2x_2 - 3x_3$$



Lower triangle

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_1 - 2x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \\ -3x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix}.$$

4. Fill in the blanks.

   (a) If $A \in \mathbb{R}^{M \times N}$, $M < N$, and $A\vec{x} = 0$ does not have a non-trivial solution, how many pivot columns does A have? $\boxed{}$

   (b) Consider the following linear transformation.

$$T(x_1, x_2) = (2x_1 - x_2,\ 4x_1 - 2x_2,\ x_2 - 2x_1)\,.$$

   The domain of T is $\boxed{}$. The image of $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under $T(\vec{x})$ is $\begin{pmatrix} \end{pmatrix}$. The co-domain of T is $\boxed{}$. The range of T is:

5. Four points in $\mathbb{R}^2$ with coordinates (t, y) are $(0, 1)$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{3}{4}, -\frac{1}{2})$. Determine the values of c_1 and c_2 for the curve $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$ that best fits the points. Write the values you obtain for c_1 and c_2 in the boxes below.

$$c_1 = \boxed{} \qquad c_2 = \boxed{}$$

In-Class Final Exam Review Set B, Math 1554, Fall 2019

1. Indicate whether the statements are true or false.

true	false	
○	○	For any vector $\vec{y} \in \mathbb{R}^2$ and subspace W, the vector $\vec{v} = \vec{y} - \text{proj}_W \vec{y}$ is orthogonal to W.
○	○	If A is $m \times n$ and has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^m$.
○	○	If a matrix is invertible it is also diagonalizable.
○	○	If E is an echelon form of A, then $\text{Null}\, A = \text{Null}\, E$.
○	○	If the SVD of $n \times n$ singular matrix A is $A = U\Sigma V^T$, then $\text{Col}A = \text{Col}U$.
○	○	If the SVD of $n \times n$ matrix A is $A = U\Sigma V^T$, $r = \text{rank}A$, then the first r columns of V give a basis for $\text{Null}A$.

2. Give an example of:

   a) a vector $\vec{u} \in \mathbb{R}^3$ such that $\text{proj}_{\vec{p}}\vec{u} = \vec{p}$, where $\vec{u} \neq \vec{p}$, and $\vec{p} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$: $\quad \vec{u} = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

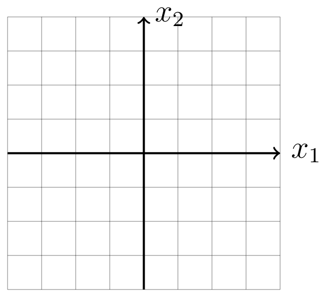
   b) an upper triangular 4×4 matrix A that is in RREF, 0 is its only eigenvalue, and its corresponding eigenspace is 1-dimensional. $\quad A = \begin{pmatrix} \quad & \quad & \quad & \quad \\ \\ \\ \\ \end{pmatrix}$

   c) A 3×4 matrix, A, and $\text{Col}(A)^\perp$ is spanned by $\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$.

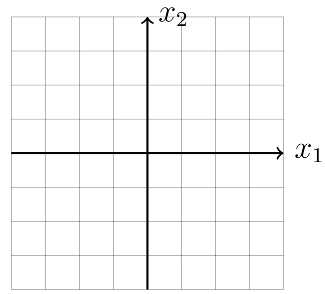
   d) A 2×2 matrix in RREF that is diagonalizable and not invertible.

3. Suppose $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$. On the grid below, sketch a) the range of $x \to Ax$, b) $(\mathrm{Col}\, A)^{\perp}$, (c) set of solutions to $A\vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

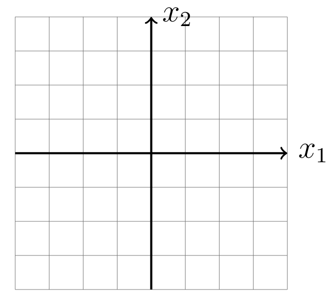
(a) range



(b) $(\mathrm{Col}\, A)^{\perp}$

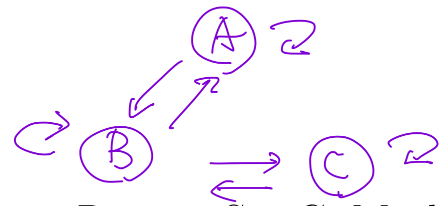


(c) solutions



4. Matrix A is a 2×2 matrix whose eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 1$, and whose corresponding eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Calculate

   1. $A(\vec{v}_1 + 4\vec{v}_2)$
   2. A^{10}
   3. $\lim\limits_{k \to \infty} A^k(\vec{v}_1 + 4\vec{v}_2)$



In-Class Final Exam Review Set C, Math 1554, Fall 2019

1. Indicate whether the statements are possible or impossible.

possible	impossible	
○	○	$Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a positive definite quadratic form, and $Q(\vec{v}) = 0$, where $\vec{v}$ is an eigenvector of A.
○	✓	The maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
✓	○	The location of the maximum value of $Q(\vec{x}) = ax_1^2 + bx_2^2 + cx_3^2$, where $a > b > c$, for $\vec{x} \in \mathbb{R}^3$, subject to $\|\|\vec{x}\|\| = 1$, is not unique.
○	○	A is 2×2, the algebraic multiplicity of eigenvalue $\lambda = 0$ is 1, and $\dim(\mathrm{Col}(A)^\perp)$ is equal to 0.
○	○	Stochastic matrix P has zero entries and is regular.
○	○	A is a square matrix that is not diagonalizable, but A^2 is diagonalizable.
○	○	The map $T_A(\vec{x}) = A\vec{x}$ is one-to-one but not onto, A is $m \times n$, and $m < n$.

$\frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = P$

P^k has no zero entries for some k.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

2. Transform $T_A = A\vec{x}$ reflects points in $\mathbb{R}^2$ through the line $y = 2 + x$. Construct a standard matrix for the transform using homogeneous coordinates. Leave your answer as a product of three matrices.

3. Fill in the blanks.

   (a) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$. What is the value of $\det(A)$? $\boxed{}$

   (b) B and C are square matrices with $\det(BC) = -5$ and $\det(C) = 2$. What is the value of $\det(B)\det(C^4)$? $\boxed{}$

   (c) A is a 6×4 matrix in RREF, and $\text{rank}(A) = 4$. How many different matrices can you construct that meet these criteria? $\boxed{}$

   (d) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, projects points onto the line $x_1 = x_2$. What is an eigenvalue of A equal to? $\boxed{}$

   (e) If an eigenvalue of A is $\frac{1}{3}$, what is one eigenvalue of A^{-1} equal to? $\boxed{}$

   (f) If A is 30×12 and $A\vec{x} = \vec{b}$ has a unique least squares solution $\hat{x}$ for every $\vec{b}$ in $\mathbb{R}^{30}$, the dimension of Null$A$ is $\boxed{}$.

4. A is a 2×2 matrix whose nullspace is the line $x_1 = x_2$, and $C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Sketch the nullspace of $Y = AC$.

5. Construct an SVD of $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Use your SVD to calculate the condition number of A.