Practice Problems for Exam 2 (Combined)

MATH 3215, Spring 2024

Chapter 3

1. The PDF of Y is given by $f_Y(y) = \frac{c}{y^3}$ for $1 < y < \infty$ and otherwise 0.
   (a) Find the value of c so that f_Y is a PDF.
   (b) Compute $\mathbb{E}[Y]$.

(a) $\int_1^\infty f_Y(y)\,dy = \int_1^\infty \frac{c}{y^3}\,dy = \left[-\frac{c}{2}y^{-2}\right]_1^\infty = \frac{c}{2} = 1$

$c = 2$

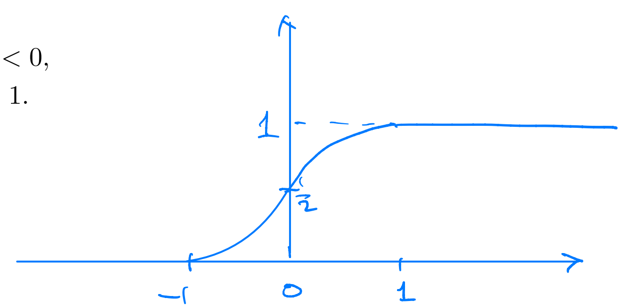
(b) $\mathbb{E}[Y] = \int_1^\infty y \cdot f_Y(y)\,dy = 2\int_1^\infty y^{-2}\,dy$

$= 2\left[-y^{-1}\right]_1^\infty = 2.$

2. Let X be a continuous random variable with the PDF

$$f_X(x) = \begin{cases} x+1, & -1 < x < 0, \\ 1-x, & 0 \le x < 1. \end{cases}$$

Find the CDF of X. Draw the graph of the CDF.



$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

If $x \le -1$, $F_X(x) = 0$

If $-1 < x < 0$, $F_X(x) = \int_{-1}^{x} (t+1)\,dt = \frac{1}{2}(x+1)^2$

If $0 \le x < 1$, $F_X(x) = \frac{1}{2} + \int_0^x (1-t)\,dt = \frac{1}{2} + x - \frac{1}{2}x^2$

$= 1 - \frac{1}{2}(x-1)^2$

If $x \ge 1$, $F_X(x) = 1$

3. Let X be the total amount of medical claims (in \$ 100,000) of the employees of a company. Assume that the PDF $f(x) = 30x(1-x)^4$ for $0 < x < 1$.

   (a) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

   (b) Find the probability that the total exceeds \$20,000.

$$\mathbb{E}[X] = \int_0^1 30x^2(1-x)^4\,dx = \int_0^1 30(1-t)^2 t^4\,dt$$

$$= 30\left[\frac{1}{5}t^5 - \frac{2}{6}t^6 + \frac{1}{7}t^7\right]_0^1 = \frac{30}{105}(21-35+15) = \frac{2}{7}.$$

$$\mathbb{E}[X^2] = \int_0^1 30x^3(1-x)^4\,dx = \frac{3}{28}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{3}{28} - \frac{4}{49} = \frac{5}{196}$$

$$P\left(X > \frac{1}{5}\right) = \int_{\frac{1}{5}}^1 30x(1-x)^4\,dx = 30\int_0^{\frac{4}{5}} (1-t)t^4\,dt$$

$$= 30\left[\frac{1}{5}t^5 - \frac{1}{6}t^6\right]_0^{\frac{4}{5}} = 6\cdot\left(\frac{4}{5}\right)^5 - 5\cdot\left(\frac{4}{5}\right)^6$$

$$= 2\left(\frac{4}{5}\right)^5$$

4. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x}$$

for $0 \le x < \infty$. Find the expected lifetime of such a tube.

$$\mathbb{E}[X] = \int_0^\infty x^2 e^{-x}\,dx = \Gamma(3) = 2! = 2.$$

5. Let X be the number of alpha particle emissions of carbon-14 that are counted by a Geiger counter each second. Assume that the distribution of X is Poisson with mean 5. Let W be the time in seconds before the third count is made.

   (a) What is the distribution of W?

   (b) Find $\mathbb{P}(W \leq 1)$.

(a) Since X is Poisson and W is the waiting time for the 2^{nd} occurrence,

$$W \sim \text{Gamma}(\underset{\alpha}{\overset{\shortparallel}{2}}, \underset{1/5}{\overset{\shortparallel}{\lambda}}).$$

(b) $\mathbb{P}(W \leq 1) = \mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1)$

↑ because $W \leq 1$ means in the time interval $[0, 1]$ at least twice happen.

$\frac{\lambda^k}{k!} e^{-\lambda}$

$$= 1 - e^{-\frac{1}{5}} - \frac{1}{5} e^{-\frac{1}{5}}$$

6. A loss (in \$ 100,000) due to fire in a building has a PDF $f(x) = \frac{1}{6}e^{-x/6}$, $0 < x < \infty$. Find the conditional probability that the loss is greater than 8 given that it is greater than 5.

Since $X \sim \text{Exp}(\frac{1}{6})$ and $F_X(x) = 1 - e^{-\frac{x}{6}}$,

$$\mathbb{P}(X > 8 \mid X > 5) = \frac{\mathbb{P}(X > 8)}{\mathbb{P}(X > 5)}$$

$$= \frac{1 - F_X(8)}{1 - F_X(5)} = \frac{e^{-\frac{8}{6}}}{e^{-\frac{5}{6}}} = e^{-\frac{1}{2}}$$

7. Let $X \sim N(650, 400)$.

(a) Find $\mathbb{P}(600 \le X < 660)$.

(b) Find a constant $c > 0$ such that $\mathbb{P}(|X - 650| \le c) = 0.95$.

(a) $Z = \dfrac{X - 650}{20} \sim N(0,1)$

$$\mathbb{P}(600 \le X < 660) = \mathbb{P}\left(\frac{600-650}{20} \le Z < \frac{660-650}{20}\right)$$

$$= \mathbb{P}(-2.5 \le Z < 0.5)$$

$$= \Phi(0.5) - \Phi(-2.5) = \Phi(0.5) + \Phi(2.5) - 1$$

(b) $\mathbb{P}(|X - 650| \le c) = \mathbb{P}\left(|Z| \le \frac{c}{20}\right)$

$$= 2\,\Phi\left(\frac{c}{20}\right) - 1 = 0.95$$

$\Phi\left(\frac{c}{20}\right) = 0.975$

By the table, $\frac{c}{20} = 1.96$, $c = 39.2$

8. Let $X \sim N(0, 4)$ and $W = X^2$. Find the PDF of W.

For $x \geq 0$, $F_W(x) = \mathbb{P}(W \le x) = \mathbb{P}(-\sqrt{x} \le X \le \sqrt{x})$

$$= F_X(\sqrt{x}) - F_X(-\sqrt{x})$$

$$f_W(x) = f_X(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} - f_X(-\sqrt{x})\left(-\frac{1}{2\sqrt{x}}\right)$$

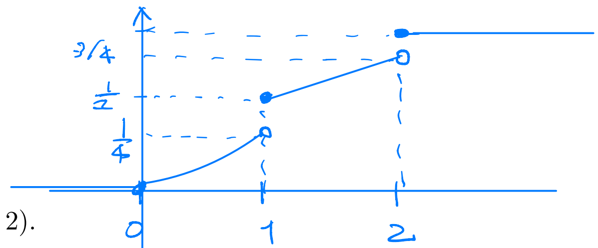
$$= \frac{1}{\sqrt{x}} f_X(\sqrt{x}) \quad \left(\text{because } f_X(\sqrt{x}) = f_X(-\sqrt{x})\right)$$

$$= \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot e^{-\frac{x}{8}}$$

If $x < 0$, $F_W(x) = 0$, $f_W(x) = 0$.

9. Let X be a random variable with the CDF

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{4}, & 0 \le x < 1, \\ \frac{x+1}{4}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases}$$



Find $\mathbb{E}[X], \mathrm{Var}(X), \mathbb{P}(\frac{1}{4} < X < 1), \mathbb{P}(X = 1), \mathbb{P}(X = \frac{1}{2}), \mathbb{P}(\frac{1}{2} \le X < 2)$.

$$\mathbb{E}[X] = \int_0^1 x \cdot \frac{x}{2}\,dx + 1 \cdot \mathbb{P}(X=1) + \int_1^2 x \cdot \frac{1}{4}\,dx + 2 \cdot \mathbb{P}(X=2)$$

$$= \frac{1}{6} + \frac{1}{4} + \frac{3}{8} + \frac{1}{2} = \frac{1}{24} \cdot (4+6+9+12) = \frac{31}{24}.$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 \cdot \frac{x}{2}\,dx + 1^2 \cdot \mathbb{P}(X=1) + \int_1^2 x^2 \cdot \frac{1}{4}\,dx + 2^2 \cdot \mathbb{P}(X=2)$$

$$= \frac{1}{8} + \frac{1}{4} + \frac{3}{12} + 1 = \frac{1}{8}(1+2+2+8) = \frac{13}{8}$$

$$\mathrm{Var}(X) = \frac{13}{8} - \left(\frac{31}{24}\right)^2$$

$$\mathbb{P}\left(\frac{1}{4} < X < 1\right) = \frac{1}{4} - \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 = \frac{15}{64}, \quad \mathbb{P}(X=1) = \frac{1}{4}, \quad \mathbb{P}(X=\tfrac{1}{2}) = 0$$

$$\mathbb{P}\left(\frac{1}{2} \le X < 2\right) = \frac{3}{4} - \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 = \frac{11}{16}.$$

10. A loss X on a car has an exponential distribution with a mean of \$5000. If the loss X on a car is greater than the deductible of \$500, the difference $X - 500$ is paid to the owner of the car. Considering zero (if $X < 500$) as a possible payment, find the expectation of the payment.

$$\text{Payment} = u(X) = \begin{cases} X - 500, & \text{if } X \ge 500 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[u(X)] = \int_{500}^{\infty} (x - 500)\frac{1}{5000} e^{-\frac{x}{5000}}\,dx$$

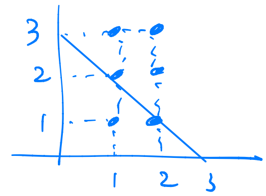
$$= 5000 \cdot \int_{\frac{1}{10}}^{\infty} \left(t - \frac{1}{10}\right) e^{-t}\,dt \quad \left(\frac{x}{5000} = t\right)$$

$$= 5000 \cdot \int_0^{\infty} u\, e^{-u - \frac{1}{10}}\,du \quad \left(t - \frac{1}{10} = u\right)$$

$$= 5000 \cdot e^{-\frac{1}{10}}.$$

Chapter 4

1. Let X and Y be discrete random variables with joint PMF



$$f(x,y) = \frac{x+y}{21}, \qquad x = 1,2, \quad y = 1,2,3.$$

   (a) Find the marginal PMFs of X and Y.

   (b) Find $\mathbb{P}(X+Y \leq 3)$.

$$f_X(x) = \sum_{y=1}^{3} f(x,y) = \frac{1}{21}(3x+6) = \frac{x+2}{7}$$

$$f_Y(y) = \sum_{x=1}^{2} f(x,y) = \frac{1}{21}(3+2y)$$

$$\mathbb{P}(X+Y \leq 3) = f(1,1) + f(1,2) + f(2,1)$$

$$= \frac{1}{21}\left((1+1) + (1+2) + (2+1)\right) = \frac{8}{21}$$

2. Let X and Y be discrete random variables with joint PMF

$$f(0,0) = f(1,2) = 0.2, \quad f(0,1) = f(1,1) = 0.3.$$

   (a) Find $\mathrm{Cov}(X,Y)$.

   (b) Find the least square regression line.

$f_X(0) = f_X(1) = \frac{1}{2}$ $\qquad$ $f_Y(0) = f_Y(2) = 0.2$, $f_Y(1) = 0.6$

$\mu_X = 0 \cdot f_X(0) + 1 \cdot f_X(1) = \frac{1}{2}$, $\quad \mu_Y = 0 \cdot f_Y(0) + 2 \cdot f_Y(2) + 1 \cdot f_Y(1)$
$= 1$

$\mathbb{E}[X^2] = 0^2 \cdot f_X(0) + 1^2 \cdot f_X(1) = \frac{1}{2}$, $\quad \mathbb{E}[Y^2] = 0^2 \cdot f_Y(0) + 2^2 \cdot f_Y(2) + 1^2 f_Y(1)$
$= 1.4$

$\sigma_X^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$, $\quad \sigma_Y^2 = 1.4 - 1^2 = 0.4$

$\mathbb{E}[XY] = \sum xy\, f(x,y) = 1 \cdot 2 \cdot 0.2 + 1 \cdot 1 \cdot 0.3 = 0.7$

$\mathrm{Cov}(X,Y) = 0.7 - \mu_X \mu_Y = 0.2$

$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = 2 \cdot \sqrt{\frac{5}{2}} \cdot 0.2 = \sqrt{\frac{2}{5}}$

$y = \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) + \mu_Y$

$= \sqrt{\frac{2}{5}} \cdot \sqrt{\frac{2}{5}} \cdot 2\left(x - \frac{1}{2}\right) + 1$

$= \frac{4}{5}x + \frac{3}{5}$

3. Let X and Y be two discrete random variables with joint PMF

$$p_{X,Y}(1,1) = p_{X,Y}(1,2) = \frac{1}{3}, \qquad p_{X,Y}(2,1) = p_{X,Y}(2,2) = \frac{1}{6}.$$

Find $\text{Cov}(X,Y)$ and ρ. Are they independent?

$$f_X(1) = \frac{2}{3} \qquad f_X(2) = \frac{1}{3} \qquad f_Y(1) = f_Y(2) = \frac{1}{2}$$

$$\mu_X = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \mu_Y = \frac{1}{2}(1+2) = \frac{3}{2}$$

$$\sigma_X^2 = 2 - \frac{16}{9} = \frac{2}{9} \qquad \sigma_Y^2 = \frac{5}{2} - \frac{9}{4} = \frac{1}{4}$$

$$\mathbb{E}[XY] = 1\cdot 1\cdot\frac{1}{3} + 1\cdot 2\cdot\frac{1}{3} + 2\cdot 1\cdot\frac{1}{6} + 2\cdot 2\cdot\frac{1}{6} = \frac{1}{6}\cdot(2+4+2+4)$$
$$= 2$$

$$\text{Cov}(X,Y) = 2 - \frac{4}{3}\cdot\frac{3}{2} = 0 = \rho.$$ Indep.

4. Let X be an exponential random variable with parameter 1, i.e., its PDF is given by $f_X(x) = e^{-x}$, $x > 0$. Find the PDF of $Y = X^4$. Are X and $Y^{\frac{1}{2}}$ negatively correlated?

For $y > 0$, $\quad F_Y(y) = P(Y \le y) = P(X \le y^{\frac{1}{4}})$
$$= F_X(y^{\frac{1}{4}})$$

$$f_Y(y) = f_X(y^{\frac{1}{4}})\cdot\frac{1}{4}\cdot y^{-\frac{3}{4}}$$
$$= \frac{1}{4} y^{-\frac{3}{4}} e^{-y^{\frac{1}{4}}}.$$

$$\text{Cov}(X, Y^{\frac{1}{2}}) = \mathbb{E}[X\cdot Y^{\frac{1}{2}}] - \mathbb{E}[X]\,\mathbb{E}[Y^{\frac{1}{2}}]$$
$$= \mathbb{E}[X^3] - \mathbb{E}[X]\,\mathbb{E}[X^2]$$

$$\mathbb{E}[X^k] = \int_0^\infty x^k e^{-x}\,dx = \Gamma(k+1) = k!$$

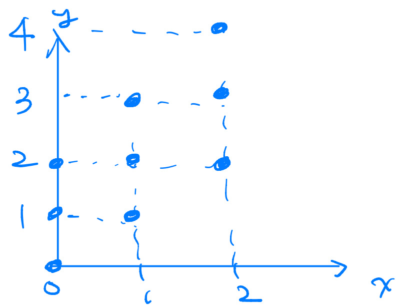
$$\text{Cov}(X, Y^{\frac{1}{2}}) = 3! - 1!\cdot 2! = 4 > 0.$$

Positively correlated.

5. Let X and Y be discrete random variables with joint PMF $f(x,y) = \frac{1}{9}$ for $(x,y) \in S$ where

$$S = \{(x,y) : 0 \leq x \leq 2, x \leq y \leq x+2, x, y \text{ are integers}\}.$$

Find $f_X(x), f_{Y|X}(y|1), \mathbb{E}[Y|X=1]$, and $f_Y(y)$.



$f_X(x) = \frac{1}{3}$ for $x = 0, 1, 2$

$f_{Y|X}(y|1) = \frac{1}{3}$ for $y = 1, 2, 3$

$\mathbb{E}[Y|X=1] = 2$

$$f_Y(y) = \begin{cases} \frac{1}{9} & \text{for } y = 0, 4 \\ \frac{2}{9} & \text{for } y = 1, 3 \\ \frac{3}{9} & \text{for } y = 2 \end{cases}$$

6. Suppose that X has a geometric distribution with parameter p. and suppose the conditional distribution of Y, given $X = x$, is Poisson with mean x. Find $\mathbb{E}[Y]$ and $\text{Var}(Y)$.

$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[X] = \frac{1}{p}$

$\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$

$= \mathbb{E}[X] + \text{Var}(X)$

$= \frac{1}{p} + \frac{1-p}{p^2} = \frac{1}{p^2}$

7. Let X and Y be two discrete random variables with joint PMF $p_{X,Y}(x,y) = \frac{x+y}{27}$, where $x = 2,3$ and $y = 1,2,3$. Find the marginal PMFs of X. Find the conditional expectation of X given $Y = 2$.

$$f_X(x) = \sum_{y=1}^{3} \frac{x+y}{27} = \frac{1}{27}(3x+6) = \frac{x+\cancel{3}}{9}$$

$$f_Y(y) = \sum_{x=2}^{3} \frac{x+y}{27} = \frac{1}{27}(5+2y)$$

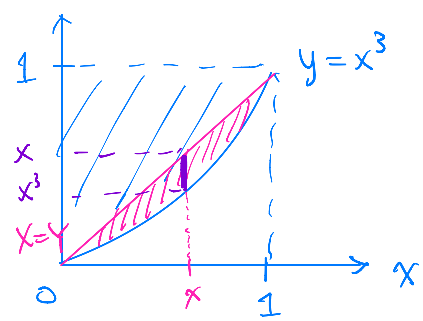
$$f_{X|Y}(x|2) = \left(\frac{x+2}{27}\right)\cdot 3 = \frac{x+2}{9}$$

$$\mathbb{E}[X \mid Y=2] = \sum_{x=2}^{3} x \cdot \frac{(x+2)}{9} = \frac{1}{9}(2\cdot 4 + 3\cdot 5) = \frac{23}{9}$$

8. Let X and Y have the joint PDF

$$f(x,y) = \frac{4}{3}, \qquad 0 < x < 1, \quad x^3 < y < 1$$

and 0 otherwise. Find $\mathbb{P}(X > Y)$.



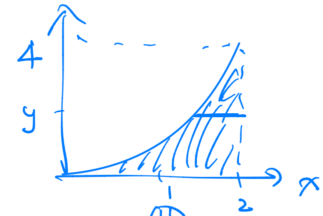
$$\mathbb{P}(X>Y) = \int_0^1 \int_{x^3}^{x} \frac{4}{3}\, dy\, dx = \frac{4}{3}\int_0^1 (x - x^3)\, dx$$

$$= \frac{4}{3}\cdot\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{3}.$$

9. Let X be a uniform random variable over $(0, 2)$ and Y given $X = x$ be $U(0, x^2)$.

   (a) Find the joint PDF $f(x, y)$ and marginal $f_Y(y)$.

   (b) Find $\mathbb{E}[Y|X]$ and $\mathbb{E}[X|Y]$.



$$f(x,y) = f_{Y|X}(y|x) \cdot f_X(x)$$

$$= \frac{1}{x^2} \cdot \frac{1}{2} \quad \text{for} \quad \underbrace{0 < y < x^2, \quad 0 < x < 2}_{\Uparrow}$$

$$f_Y(y) = \int_{\sqrt{y}}^{2} f(x,y)\,dx = \frac{1}{2}\left[-\frac{1}{x}\right]_{\sqrt{y}}^{2} = \frac{1}{2}\left(\frac{1}{\sqrt{y}} - \frac{1}{2}\right)$$

$$E[Y|X] = \frac{1}{2}X^2$$

$$f_{X|Y}(x|y) = \frac{1}{x^2} \cdot \left(\frac{1}{\sqrt{y}} - \frac{1}{2}\right)^{-1} \quad \text{for} \quad \sqrt{y} \le x \le 2$$

$$E[X|Y=y] = \int_{\sqrt{y}}^{2} x \cdot \frac{1}{x^2} \cdot \left(\frac{1}{\sqrt{y}} - \frac{1}{2}\right)^{-1} dx$$

$$= \left(\frac{1}{\sqrt{y}} - \frac{1}{2}\right)^{-1} \cdot \ln\frac{2}{\sqrt{y}}$$

$$E[X|Y] = \ln\left(\frac{2}{\sqrt{Y}}\right) \cdot \left(\frac{2\sqrt{Y}}{2 - \sqrt{Y}}\right).$$

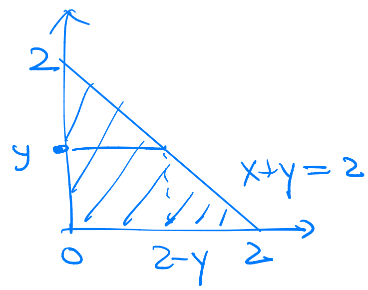
10. Let X and Y be two random variables with joint PDF $f_{X,Y}(x, y) = 2e^{-x-y}$ for $0 \le x \le y < \infty$. Find the marginal PDFs of X and Y. Are they independent?

$$f_X(x) = \int_{x}^{\infty} 2e^{-x-y}\,dy = 2e^{-2x}, \quad x \ge 0$$

$$f_Y(y) = \int_{0}^{y} 2e^{-x-y}\,dx = 2e^{-y}(1 - e^{-y}), \quad y \ge 0$$

$$f_X(x) \cdot f_Y(y) \ne f(x,y) \qquad \text{Not Indep.}$$

11. Let X and Y be two random variables with joint PDF $f_{X,Y}(x, y) = \frac{1}{2}$ for $0 < x + y < 2$, $x > 0$, $y > 0$. Find the conditional expectation $\mathbb{E}[X|Y]$.



$$f_Y(y) = \int_0^{2-y} \frac{1}{2}\, dx = \frac{1}{2}(2-y)$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{2-y}$$

$$\mathbb{E}[X|Y=y] = \int_0^{2-y} \frac{1}{2-y} \cdot x\, dx = \frac{2-y}{2} = 1 - \frac{y}{2}$$

$$\mathbb{E}[X|Y] = 1 - \frac{Y}{2}.$$

12. Let (X, Y) be a bivariate normal random vector. Both X and Y have mean 1 and variance 4, while the correlation coefficient of X and Y is $\rho = \frac{1}{4}$. Find $\mathrm{Var}(-2X + Y)$. Assume now that (X, Y) is a bivariate normal vector, with X and Y having mean 9 and variance 9, but that the correlation coefficient has changed and is now given by $\rho = 0$. Find $\mathbb{P}(3 \leq X \leq 15, Y \leq 9)$ and using the tables, find an approximate value for it.

$$\mathrm{Var}(-2X+Y) = 4\,\mathrm{Var}(X) - 4\,\mathrm{Cov}(X,Y) + \mathrm{Var}(Y)$$

$$= 4 \cdot 4 - 4 \cdot \rho \cdot \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} + 4$$

$$= 16 - 4 \cdot \frac{1}{4} \cdot 4 + 4 = 16.$$

(X, Y) Bivariate Normal and $\rho = 0$ implies

X, Y are indep.

$$\Rightarrow \quad \mathbb{P}(3 \leq X \leq 15, Y \leq 9)$$

$$= \mathbb{P}(3 \leq X \leq 15)\, \mathbb{P}(Y \leq 9)$$

$$= \mathbb{P}\left(\frac{3-9}{3} \leq Z \leq \frac{15-9}{3}\right) \mathbb{P}\left(Z \leq \frac{9-9}{3}\right)$$

$$= (\Phi(2) - \Phi(-2)) \cdot \Phi(0) = \frac{1}{2}(2\Phi(2) - 1)$$

13. The life of a certain type of automobile tire is normally distributed with mean 34,000 miles and standard deviation 4000 miles.

    (a) What is the probability that such a tire lasts more than 40,000 miles?

    (b) What is the probability that it lasts between 30,000 and 35,000 miles?

    (c) Given that it has survived 30,000 miles, what is the conditional probability that the tire survives another 10,000 miles?

X = The lifetime of a tire $\sim N(\underbrace{34{,}000}_{=\mu}, \underbrace{4000^2}_{=\sigma^2})$

(a) $P(X > 40{,}000) = P\left(\frac{X-\mu}{\sigma} > \frac{40{,}000 - 34{,}000}{4000}\right)$

$= 1 - \Phi(1.5) = 1 - 0.9332 = 0.0668$

(b) $P(30{,}000 < X < 35{,}000)$

$= P\left(\frac{30{,}000 - 34{,}000}{4000} < \frac{X-\mu}{\sigma} < \frac{35{,}000 - 34000}{4000}\right)$

$= \Phi(0.25) - \Phi(-1) = \Phi(0.25) + \Phi(1) - 1 = 0.5987 + 0.8413 - 1$

(c) $P(X > 40{,}000 \mid X > 30{,}000) = \frac{P(X > 40000)}{P(X > 30000)} = \frac{0.0668}{0.8413}$

~~14.~~ A certain type of electrical motors is defective with probability 1/100. Pick 1000 motors and let X be the number of defective ones among these 1000 motors. Using a normal approximation, (with/without) mid-point correction, write down an expression for the probability that among the 1000 motors 13 or less are defective.

Chapter 5

1. Let X be a uniform random variable on $(-1,3)$ and $Y = X^2$. Find the PDF of Y.

Case 1 : $y < 0$, $F_Y(y) = \mathbb{P}(Y \leq y) = 0$

Case 2 : $0 \leq y < 1$ $F_Y(y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = \frac{1}{4} \cdot 2\sqrt{y} = \frac{\sqrt{y}}{2}$

Case 3 : $1 \leq y < 9$ $F_Y(y) = \mathbb{P}(-1 \leq X \leq \sqrt{y}) = \frac{1}{4}(\sqrt{y}+1)$

Case 4 : $y > 9$ $F_Y(y) = 1$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}}, & 0 \leq y < 1 \\ \frac{1}{8\sqrt{y}}, & 1 \leq y < 9 \\ 0, & \text{otherwise.} \end{cases}$$

2. Let X be a random variable with PDF given by

$$f_X(x) = x^2 + \frac{10x^4}{3}$$

for $0 < x < 1$, and otherwise $f_X(x) = 0$. Find the CDF and the PDF of $Y = \log X$.

Since $0 < X < 1$, $-\infty < Y = \log X < 0$. For $y < 0$,

$$F_Y(y) = P(Y \leq y) = P(\log X \leq y)$$
$$= \mathbb{P}(X \leq e^y) = F_X(e^y)$$

$$f_Y(y) = f_X(e^y) \cdot e^y = \left(e^{2y} + \frac{10}{3} e^{4y}\right) \cdot e^y \quad \text{for } y < 0$$

otherwise 0.

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{1}{3}x^3 + \frac{2}{3}x^5 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

$$F_Y(y) = \begin{cases} \frac{1}{3}e^{3y} + \frac{2}{3}e^{5y} & \text{if } y < 0 \\ 1 & \text{if } y \geq 0 \end{cases}$$

3. Let X be a random variable with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find the density of $1/X$.

$$Y = \frac{1}{X}$$

$$F_Y(y) = P\left(\frac{1}{X} \leq y\right) = P\left(X \geqslant \frac{1}{y}\right) = 1 - F_X\left(\frac{1}{y}\right)$$

$$f_Y(y) = -f_X\left(\frac{1}{y}\right)\cdot\left(-\frac{1}{y^2}\right) = -\frac{1}{\pi\left(1+\frac{1}{y^2}\right)}\cdot\left(-\frac{1}{y^2}\right) = \frac{1}{\pi(1+y^2)}$$

$$= f_X(y)$$