Lecture 2. Basic Combinatorics II (Sec 1.4-6)

University of Illinois at Urbana–Champaign

Math 461 Spring 2022

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Recall

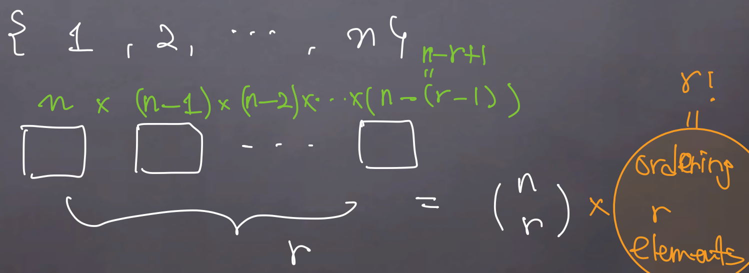
Combination

don't care the ordering.

Consider n distinct objects.
How many different groups, called combinations, of size r $(1 \leqslant r \leqslant n)$ of these objects can be formed?
The number of combinations is

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdots (n-r+1)}{r \cdot (r-1) \cdots 1} = \frac{n!}{(n-r)!r!}.$$

n choose r



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$$n \cdot (n-1) \cdot \cdots (n-r+1) = \binom{n}{r} \cdot r!$$

$$\Rightarrow \binom{n}{r} = \frac{n \cdot (n-1) \cdot \cdots (n-r+1)(n-r)(n-r-1) \cdots 2 \cdot 1}{r! \, (n-r)(n-r-1) \cdots 3 \cdot 2 \cdot 1}$$

$$= \frac{n!}{r! \, (n-r)!}$$

Combination

Example

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

$$= \binom{20}{3} = \frac{20!}{3! \cdot (20-3)!} = \frac{20 \cdot 19 \cdot \overset{3}{\cancel{18}}}{\cancel{3 \cdot 2} \cdot 1}$$

$$= \underline{\underline{20 \cdot 19 \cdot 3}}$$

Note: Replace 3 with 17, $\binom{20}{3} = \binom{20}{17}$

get the same answer.

In general, $\binom{n}{r} = \binom{n}{n-r}$

Combination

Example

Consider a set of 8 antennas of which 3 are defective and 5 are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in ~~which no two defectives are consecutive~~?

Method 1

$D_1, D_2, D_3, F_1, F_2, F_3, F_4, F_5.$

$$\frac{8!}{3! \cdot 5!} = \binom{8}{3} \leftarrow \text{why?}$$

Method 2

$\binom{8}{3}$ Done.

Combination

Example

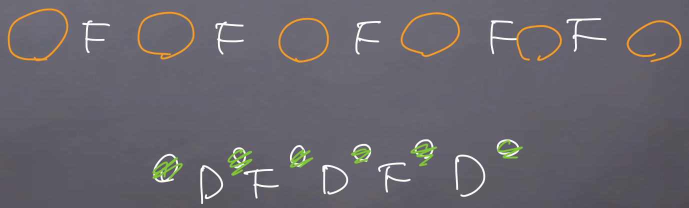
Consider a set of 8 antennas of which 5 are defective and 3 are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

$$\binom{6}{3} = \frac{6!}{3!\,(6-3)!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20.$$

Combination

Example

Consider a set of 8 antennas of which 5 are defective and 3 are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?



Binomial Theorem

For real numbers x, y and a natural number n, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

proof next time.

Ex

$n = 1$ $\quad (x+y)^1 = x+y$

$$\sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 \cdot y^{1-0} + \binom{1}{1} x^1 y^{1-1} = x + y.$$

$n=2$ $\quad (x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$

Multinomial Coefficients

For integer $n \geqslant 1$ and integers $n_1, \cdots, n_r \geqslant 0$ such that $n_1 + \cdots + n_r = n$,

$$\binom{n}{n_1, \cdots, n_r} = \frac{n!}{n_1! \cdots n_r!}.$$

Multinomial Coefficients: First Interpretation

The number of ways to divide n distinct objects into r *distinct groups* of size $n_1, \cdots, n_r$ with $n_1 + \cdots + n_r = n$ is

$$\binom{n}{n_1, \cdots, n_r}.$$

Multinomial Coefficients: First Interpretation

Example

Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible? What if we just divide them into two teams?

Multinomial Coefficients: Second Interpretation

The number of ordered arrangements of n objects of which n_1 are alike, $\cdots$, n_r are alike with $n_1, \cdots, n_r$ with $n_1 + \cdots + n_r = n$ is

$$\binom{n}{n_1, \cdots, n_r}.$$

Multinomial Coefficients: Second Interpretation

Example

How many different letter arrangements can be formed from the letters *PEPPER*?

Multinomial Theorem

For real numbers $x_1, \cdots, x_r$ and an integer $n \geqslant 1$, we have

$$(x_1 + \cdots + x_r)^n = \sum \binom{n}{n_1, \cdots, n_r} x_1^{n_1} \cdots x_r^{n_r}$$

where the sum is over all integers $n_1, \cdots, n_r \geqslant 0$ such that $n_1 + \cdots + n_r = n$.

Multinomial Theorem

Example

What is the coefficient of the term x^2y^3z in the expansion of $(x+y+z)^6$?

Multinomial Theorem

Example

Consider n distinct balls and r distinct urns. How many different ways are there to distribute balls into urns? (An urn can contain any number of balls, including zero.) What if the balls are indistinguishable?