Chapter 3. Discrete Random Variables and Probability Distributions

Math 3670 Summer 2024

Georgia Institute of Technology

Section 1.
Random Variables

Random Variables

Definition

For a given sample space S of some experiment, a random variable (RV) is any rule that associates a number with each outcome in $\mathcal{S}$.

In mathematical language, a random variable X is a function whose domain is the sample space $\mathcal{S}$ and whose range is the set of real numbers $\mathbb{R}$, that is

$$X : \mathcal{S} \to \mathbb{R}.$$

the set of real numbers

$S = \{H, T\}$

$X : S \to \mathbb{R}$

$$\begin{cases} X(H) = 1 \\ X(T) = 0 \end{cases}$$

$$\begin{cases} Y(H) = 100 \\ Y(T) = -200 \end{cases}$$

Random Variables

Example

When a student calls a university help desk for technical support, he/she will either immediately be able to speak to someone (S, for success) or will be placed on hold (F, for failure).

With $\mathcal{S} = \{S, F\}$, define an RV X by

$$X(S) = 1, \qquad X(F) = 0.$$

Bernoulli Random Variables

Definition

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

$\{$ 2 outcomes

$$X = \begin{cases} 1 \\ 0 \end{cases}$$

Examples

Example

Roll two dice.

Then, $\mathcal{S} = \{(1,1), (1,2), (1,3), \ldots, (6,6)\}$

Let

$$X = \text{The sum of two outcomes}$$
$$Y = \text{The product of two outcomes}$$
$$U = \text{The maximum of two outcomes}$$

$X = \{2, 3, \ldots, 12\}$

What are the possible values for X, Y, U?

$\mathbb{P}(X=2) = \frac{1}{36}$

$Y = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, 36\}$

$U = \{1, 2, \ldots, 6\}$

$\mathbb{P}(U=4) = \frac{7}{36}$

(1,4) (4,1)

(2,4) (4,2)

(3,4) (4,3)

(4,4)

Examples

Example

Consider an experiment in which 9-volt batteries are tested until one with an acceptable voltage is obtained.

The sample space is $\mathcal{S} = \{S, FS, FFS, \cdots\}$.

Let

$$X = \text{the number of batteries tested before the experiment terminates.}$$

$X = 1, 2, \cdots\cdots$

A Geometric RV

Bernoulli Exp.

//

- Repeat $\boxed{\text{Exp w/ 2 outcomes}}$ until success.
- $X = \#$ of trials.

Def • X is Bernoulli RV

$(X \sim Ber(p))$

success probability

$$X = \begin{cases} 1 \\ 0 \end{cases}, \quad \begin{aligned} P(X=1) &= p \in (0,1) \\ P(X=0) &= 1-p \end{aligned}$$

• X is a geomtric RV

$(X \sim Geom(p))$

X = # of trials until the first success

$$\mathbb{P}(X=2) = \mathbb{P}(\underline{F}\, S) = (1-p)\cdot p$$

Discrete Random Variables

Definition

A discrete random variable is an RV whose possible values either constitute a finite set or else can be listed in an infinite sequence in which there is a first element, a second element, and so on (“countably” infinite).

Note: We will discuss Continuous Random variables later.

By 2:58

Exercise

(3.1-4) Let X be the number of nonzero digits in a randomly selected zip code.

What are the possible values of X?

Give three possible outcomes and their associated X values.

$$X = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases}$$

$$30097 \xmapsto{X} 3$$

$$P(X=3) = \binom{5}{3} \left(\frac{9}{10}\right)^3 \cdot \left(\frac{1}{10}\right)^2$$

S S S F F

S F S F S

3S 2F

$S_1 S_2 S_3 F_1 F_2$ How many arrangement?

Assume all distinct $\Rightarrow 5!$

arr. of SSSFF

SSSFF

SFFSS

⋮

$S_1 S_2 S_3 F_1 F_2$

$S_2 S_1 S_3 F_2 F_1$

⋮

$\Big\}\ 3! \cdot 2!$

$$X \cdot 3!\,2! = 5!$$

$$X = \frac{5!}{3!\,2!} = \binom{5}{2}$$

AAA BBB CCC DDD

$$\frac{12!}{3!\,3!\,3!\,3!}$$

$\mathcal{S}$: Sample space

$\underbrace{X}_{\text{a function}} : \mathcal{S} \to \mathbb{R}$ $\triangleq$ Random Variable.

Example

An experiment with $\mathcal{S} = \{ S, F \}$

Bernoulli Exp.

$P(\underbrace{X = 1}_{S}) = p$, $P(\underbrace{X = 0}_{F}) = 1 - p$ $0 < p < 1$

Bernoulli Random Variable

Example Repeat Independent Bernoulli Exp until the first success.

X = # of trials

Geometric Random Variable.

Section 2.
Probability Distributions for Discrete Random Variables

discrete

Assume X : random variable.

Probability Mass Functions

Definition

The probability mass function (PMF) of a discrete RV X is defined for every number x by

$$p(x) = \mathbb{P}(X = x) = \mathbb{P}(\{s \in \mathcal{S} : X(s) = x\}).$$

Ex

X : Bernolli RV.

$p(1) = \mathbb{P}(X = 1)$ $\quad$ $p(0) = \mathbb{P}(X = 0)$

Probability Mass Functions

Example

Consider whether the next person buying a computer at a certain electronics store buys a laptop or a desktop model. Let

Bernoulli.

$$X = \begin{cases} 1, & \text{if the customer purchases a desktop computer} \quad = S \\ 0, & \text{if the customer purchases a laptop computer.} \quad = F \end{cases}$$

If 20% of all purchasers during that week select a desktop, the PMF for X is

$p(1) = \mathbb{P}(X=1) = P(\text{Desktop}) = 0.2$

$p(0) = P(X=0) = P(\text{laptop}) = 1 - 0.2 = 0.8$

PMF of X:

$$p(x) = \begin{cases} 0.2, & x = 1 \\ 0.8, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

X, Y are the same in terms of RV

when PMF of X = PMF of Y

Probability Mass Functions

Example

Consider selecting at random a student who is among the 15,000 registered for the current term at Mega University.

Let X be the number of courses for which the selected student is registered with the PMF

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

What is the probability that the chosen student is registered for more than 4 courses?

$$\mathbb{P}(X > 4) = \mathbb{P}(X = 5, 6, 7)$$

$$= \mathbb{P}(X=5) + \mathbb{P}(X=6) + \mathbb{P}(X=7)$$

$$= p(5) + p(6) + p(7)$$

$$= 0.39 + 0.17 + 0.02$$

A Parameter of a Probability Distribution

Example

Consider whether the next person buying a computer at a certain electronics store buys a laptop or a desktop model. Let

$$X = \begin{cases} 1, & \text{if the customer purchases a desktop computer} \\ 0, & \text{if the customer purchases a laptop computer.} \end{cases}$$

Then, X is a Bernoulli. random variable.

If $100\alpha\%$ of all purchasers during that week select a desktop, the PMF for X is

$$p(x) = \begin{cases} \alpha, & x=1 \\ 1-\alpha, & x=0 \\ 0, & \text{otherwise} \end{cases}$$

Common mistake

$$p(x) = \begin{cases} 1 \\ 0 \\ \text{otherwise} \end{cases}$$

A Parameter of a Probability Distribution

PMF of X

Definition

Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution.

Such a quantity is called a parameter of the distribution.

The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

$$X = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases}$$

parameter

$X \sim \mathrm{Ber}(p)$

A Parameter of a Probability Distribution

Example

Starting at a fixed time, we observe the gender of each newborn child at a certain hospital until a boy (B) is born.

Let $p = \mathbb{P}(B)$, assume that successive births are independent, and define the RV X by the number of births observed.

Find the PMF of X.

$p \quad 1-p$

$\{S, F\}$ repeat until 1^{st} S

X = # of trials

A Parameter of a Probability Distribution

$p(3.5) = \mathbb{P}(X = 3.5) = 0$

Definition

We say X is a geometric random variable with success probability p if its PMF is

$$p(x) = \begin{cases} (1-p)^{x-1} \cdot p & \text{if } x = 1, 2, 3, \cdots \\ 0 & \text{otherwise} \end{cases}$$

① Possible values? $1, 2, 3, \cdots\cdots$

② $p(1), \quad p(2), \quad p(3), \cdots$

$p(1) = \mathbb{P}(X=1) = p$

$p(2) = (1-p) \cdot p$

$p(3) = (1-p)^2 p$

$\vdots$

$p(k) = (1-p)^{k-1} \cdot p$

$\underbrace{F\, F \cdots F}_{(k-1) \text{ times}}\, S$

<u>Note</u> $X \sim \text{Geom}(\underline{p})$ ← success prob.

$$p(1) + p(2) + p(3) + \cdots\cdots = \underline{1}$$

$$p + (1-p)\cdot p + (1-p)^2\cdot p + (1-p)^3\cdot p + \cdots = \frac{\overset{\text{1st term}}{p}}{1-\underbrace{(1-p)}_{\text{Ratio}}}$$

$$= \frac{p}{p} = 1.$$

$$P(X \text{ is even})$$

$$= p(2) + p(4) + p(6) + \cdots$$

$$= (1-p)\cdot p + (1-p)^3\cdot p + \cdots = \frac{p(1-p)}{1-(1-p)^2} = \frac{1-p}{2-p}$$

The Cumulative Distribution Function

PMF : $p(x) = \mathbb{P}(X = x)$

← (only for discrete)

Definition

The cumulative distribution function (CDF) $F(x)$ of a discrete RV variable X with PMF $p(x)$ is defined by

$$F(x) = \mathbb{P}(X \leq x)$$

↖ (For any RV)

$p(3.5) = \mathbb{P}(X=3.5) = 0$

The Cumulative Distribution Function

Example

Starting at a fixed time, we observe the gender of each newborn child at a certain hospital until a boy (B) is born.

Let $p = \mathbb{P}(B)$, assume that successive births are independent, and define the RV X by the number of births observed.

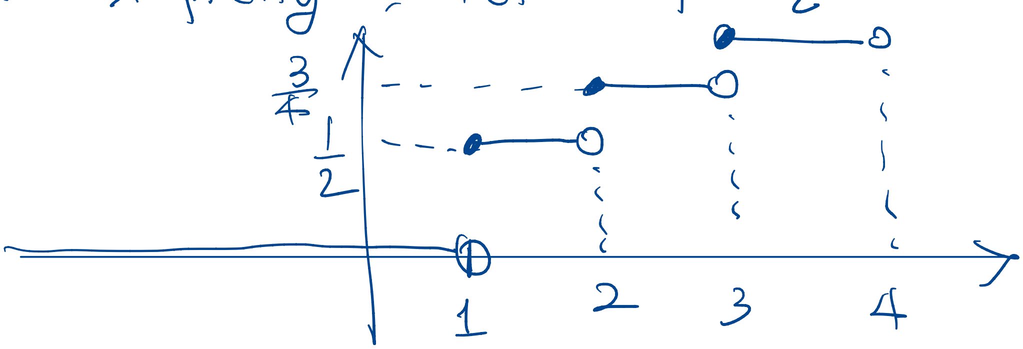
Find the CDF.

$X \sim \text{Geom}(p)$

$$F(x) = \mathbb{P}(X \le x)$$

$$F(2) = \mathbb{P}(X \le 2) = \mathbb{P}(X = 1, 2)$$
$$= p(1) + p(2) = p + (1-p)\cdot p$$

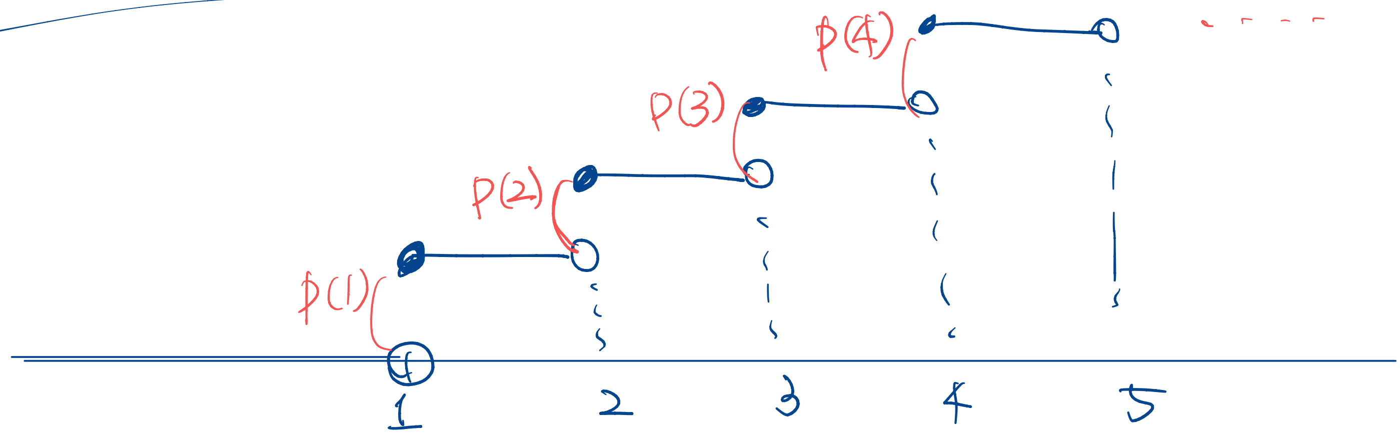
For simplicity, let $p = \frac{1}{2}$



$F(1) = \frac{1}{2}$

$F(1.7)$

CDF of F



Exercise

By 3:02

(3.2-17) A new battery's voltage may be acceptable (A) or unacceptable (U).

A certain flashlight requires two batteries, so batteries will be independently selected and tested until two acceptable ones have been found.

Suppose that 90% of all batteries have acceptable voltages.

Let Y denote the number of batteries that must be tested.

Find $p(2)$, $p(3)$, and $p(5)$.

① until 5 success

② consecutive 2 success

$Y=2$: AA

$Y=3$: UAA , AUA

$Y=5$: $\underbrace{\square\,\square\,\square\,\square}_{1A\ \ 3U}$ A $\binom{4}{1}$

$Y=k$ $\overbrace{\boxed{1A \quad (k-2)\,U}}^{(k-1)\text{ spots}}$ A $\binom{k-1}{1}$

$$p(k) = P(Y=k) = \binom{k-1}{1} p^2 (1-p)^{k-2}$$

Repeat until 2 Consec. S!

X = # trials.

$X = 2$: S S

$X = 3$: F S S

$X = 4$: F F S S   S F S S

$X = 5$: $\square\ \square$ F S S

F F

S F

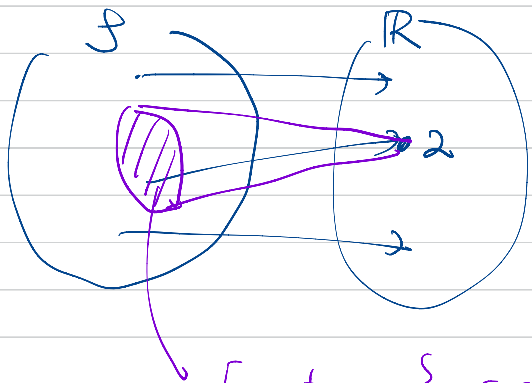
F S

$X = k$:

Recall

$X : S \to \mathbb{R}$

outcomes $\longmapsto$ numbers



$P(X=2)$

Event $\{ s \in S : X(s) = 2 \} = \{ X = 2 \}$

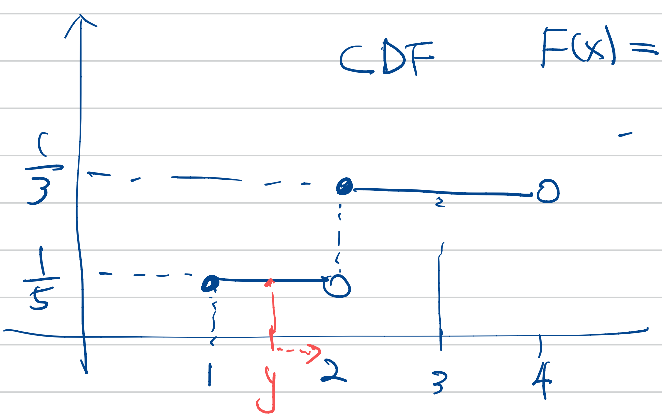
$p(x) = P(X = x)$, $x \in \mathbb{R}$

PMF from $\mathbb{R}$ to $[0, 1]$

$F(x) = P(X \le x)$, $x \in \mathbb{R}$

Cumulative Distribution Function.

CDF $F(x) = P(X \le x)$



$F(3) = \frac{1}{3} = P(X \le 3)$

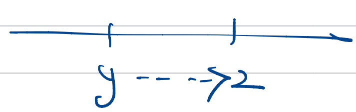
$P(X < 2) = \frac{1}{5}$

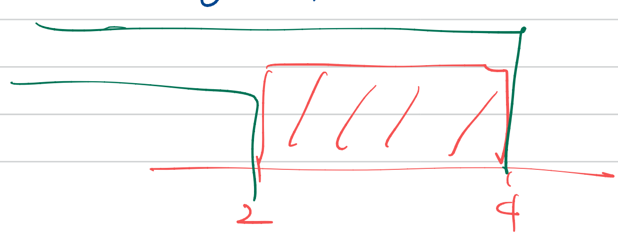
$= \lim_{y \uparrow 2} P(X \le y) = \lim_{y \uparrow 2} F(y)$

$P(2 < X < 4)$

$= P(X < 4) - P(X \le 2)$

$= \lim_{y \uparrow 4} F(y) - F(2) = 0$





Section 3.
Expected Values

& Variance of a RV.

Expected Values

Consider a university having 15,000 students and let X be the number of courses for which a randomly selected student is registered.

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02
number registerd	150	450	1950	3750	5850	2550	300

The average number of courses per student is

$$p(x) = P(X = x) = \frac{\text{\# of std registered } x \text{ Courses}}{15,000}$$

$$\frac{\text{\# of the courses registered}}{\text{\# of std.}}$$

$$= \frac{1 \cdot 150 + 2 \cdot 450 + 3 \cdot 1950 + \cdots}{15000}$$

$$= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3) + \cdots + 7 \cdot p(7)$$

$$= \sum x\, p(x) \leftarrow \text{Expected \# of Courses.}$$

Expected Values

Definition

Let X be a discrete RV with set of possible values D and PMF $p(x)$.

The expected value or mean value of X is

Expectation

$$\mathbb{E}[X] = \mu_X = \mu = \sum_{x \in D} x \cdot p(x)$$

Expected Values

Example

Let X be a Bernoulli random variable with p.

Find the expectation of X.

success probability

Recall

$X \sim Ber(p)$

$$X = \begin{cases} 1 \\ 0 \end{cases} \qquad \mathbb{P}(X=1) = p, \quad \mathbb{P}(X=0) = 1-p.$$

$$\mathbb{E}[X] = \sum_{x \in D} x \cdot p(x) = 0 \cdot \underbrace{p(0)}_{= \mathbb{P}(X=0) = 1-p} + 1 \cdot \underbrace{p(1)}_{= \mathbb{P}(X=1) = p}$$

$$= p$$

Expected Values

Example

Let X be a geometric random variable with p.

Find the expectation of X.

Recall $X \sim Geom(p)$

$X = \begin{cases} 1 \\ 2 \\ \vdots \end{cases}$

$p(k) = \mathbb{P}(X = k)$

$= 1 \cdot (1-p)^{k-1} \cdot p$

= # of trials until 1st success.

$\underbrace{F\,F\,F \cdots F}_{k-1}\ S$

$$\mathbb{E}[X] = \sum_{x \in D} x \cdot p(x) = \sum_{x=1}^{\infty} x \cdot (1-p)^{x-1} \cdot p$$

$$= 1 \cdot (1-p)^{1-1} \cdot p + 2 \cdot (1-p)^{2-1} \cdot p + 3\,(1-p)^{3-1} \cdot p + \cdots$$

For simplicity,

if $p = \frac{1}{2}$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2 + 3 \cdot \left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \cdots$$

$$- \quad \frac{1}{2}\mathbb{E}[X] = 1 \cdot \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{4}\right)^4 + \cdots$$

$$\frac{1}{2}\mathbb{E}[X] = \underbrace{(1-0)\left(\frac{1}{2}\right) + (2-1)\left(\frac{1}{2}\right)^2 + (3-2)\left(\frac{1}{2}\right)^3 + \cdots}_{\text{Geometric Series}}$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \underline{1}$$

$$\mathbb{E}[X] = 2$$

$$\mathbb{E}[X] = 1 \cdot (1-p)^0 \cdot p + 2 \cdot (1-p)^1 \cdot p + 3\,(1-p)^2 \cdot p + \cdots$$

$$(1-p)\,\mathbb{E}[X] = \quad + 1 \cdot (1-p)^1 \cdot p + 2\,(1-p)^2 \cdot p + \cdots$$

$$\big(1-(1-p)\big)\,\mathbb{E}[X] = (1-p)^0 \cdot p + (1-p)^1 \cdot p + (1-p)^2 \cdot p + \cdots$$

$$p\,\mathbb{E}[X] = \frac{p}{1-(1-p)} = 1$$

$$\mathbb{E}[X] = \frac{1}{p}.$$

The Expected Value of a Function

Example

Suppose a bookstore purchases ten copies of a book at \$6.00 each to sell at \$12.00 with the understanding that at the end of a 3-month period any unsold copies can be redeemed for \$2.00.

If X is the number of copies sold, then net revenue is

$$h(X) = -6 \cdot 10 + X \cdot 12 + 2 \cdot \underline{(10 - X)}$$

What then is the expected net revenue?

$$= -60 + 12X + 20 - 2X$$

$$= 10X - 40 .$$

Expected Net Revenue.

$$= \underline{(10 \cdot 0 - 40)} \cdot p(0) + \underline{(10 \cdot 1 - 40)}\, p(1)$$

$$+ (10 \cdot 2 - 40) \cdot p(2) + \cdots + (10 \cdot 10 - 40)\, p(10)$$

$$= \sum h(x) \cdot p(x) = E[h(X)]$$

h : a function from $\mathbb{R}$ to $\mathbb{R}$

X : random variable

$$\begin{tikzcd} S \arrow[r, mapsto, "X"] \arrow[rr, bend right, mapsto] & \mathbb{R} \arrow[r, mapsto, "h"] & \mathbb{R} \end{tikzcd}$$

$h(X) \leftarrow$ another random variable.

The Expected Value of a Function

Proposition

If the RV X has a set of possible values D and PMF $p(x)$, then the expected value of any function $h(X)$ is

$$\mathbb{E}[h(X)] = \sum_{y \in \{\text{Possible values for } h(X)\}} y \cdot \mathbb{P}(h(X) = y)$$

$$= \sum_{x \in D} h(x) \cdot \mathbb{P}(X = x)$$

D = { the of all possible values for X }

Exercise

A computer store has purchased three computers of a certain type at $500 apiece.

It will sell them for $1000 apiece.

The manufacturer has agreed to repurchase any computers still unsold after a specified period at $200 apiece.

Let X denote the number of computers sold with PMF

$$p(0) = .1, \quad p(1) = .2, \quad p(2) = .3, \quad p(3) = .4.$$

Find the expected profit

Variance and Standard Deviation

Definition

Let X have PMF $p(x)$ and expected value μ.

a function of X

Then the variance of X is defined by

$$\mathrm{Var}(X) = \sigma^2 = \sigma_X^2 = \mathbb{E}\Big[\overbrace{(X - \mathbb{E}[X])^2}^{\text{a function of } X}\Big]$$

The standard deviation (SD) of X is

$$h(x) = (x - \underline{\mathbb{E}[X]})^2$$

Not Random

$$\sigma_X = \sqrt{\mathrm{Var}(X)}$$

Note

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

$$= \sum_x (x - \mathbb{E}[X])^2 \cdot p(x)$$

Variance and Standard Deviation

Example

Let X be a discrete RV with PMF

$$p(0) = .1, \quad p(1) = .2, \quad p(2) = .3, \quad p(3) = .4.$$

Find the variance.

$$\mathbb{E}[X] = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$
$$= 0.2 + 0.6 + 1.2 = 2$$

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2 \right]$$
$$= \sum_x (x - E[X])^2 \cdot p(x)$$
$$= (0-2)^2 \cdot p(0) + (1-2)^2 p(1) + (2-2)^2 p(2) + (3-2)^2 p(3)$$
$$= 4 \cdot (0.1) + 1 \cdot (0.2) + 0 \cdot (0.3) + 1 \cdot (0.4)$$
$$= 1.$$

Properties

$$\begin{cases} \bullet\ \mathbb{E}[\underbrace{c}_{\text{Non-random}}] = c \\ \bullet\ \mathbb{E}[c \cdot X] = c \cdot \mathbb{E}[X] \end{cases}$$

$\bullet\ \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Note: $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Variance and Standard Deviation

$$(x-\mu)^2 = x^2 - 2\mu \cdot x + \mu^2$$

(Shortcut Formula)

Proposition

$$\mathrm{Var}(X) = \mathbb{E}[(X - \underbrace{\mathbb{E}[X]}_{=\mu})^2]$$

$$= \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] + \mathbb{E}[(-2\mu) \cdot X] + \mathbb{E}[\mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \cdot \underbrace{\mathbb{E}[X]}_{=\mu} + \mu^2$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

2/6/25

<u>Recall</u>

X : a discrete RV with PMF $p(x)$

$$\mathbb{E}[X] = \sum x \cdot p(x)$$

$h : \mathbb{R} \to \mathbb{R}$ a function

$h(X)$: a new RV.

$$\mathbb{E}[h(x)] = \sum h(x) \cdot \underbrace{p(x)}_{\text{PMF of } X}$$

$$\operatorname{Var}(X) \overset{\text{def}}{=} \mathbb{E}[\underbrace{(X - \mathbb{E}[X])^2}_{= h(X)}] = \sum (x - \mathbb{E}[X])^2 p(x)$$

$$\underset{\text{Shortcut formula}}{=} \underline{\mathbb{E}[X^2] - (\mathbb{E}[X])^2} = \sum x^2 \cdot p(x) - \left(\sum x p(x)\right)^2$$

= (Expectation of Square) − (Square of Exp.)

Variance and Standard Deviation

Example

Let X be a discrete RV with PMF

$$p(0) = .1, \quad p(1) = .2, \quad p(2) = .3, \quad p(3) = .4.$$

Find the variance using the formula.

Known

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$

$$= 2$$

$$E[X^2] = 0^2 \cdot p(0) + 1^2 \cdot p(1) + 2^2 \cdot p(2) + 3^2 \cdot p(3)$$

$$= 0 \cdot (0.1) + 1 \cdot (0.2) + 4(0.3) + 9(0.4)$$

$$= 0.2 + 1.2 + 3.6 = 5$$

$$Var(X) = E[X^2] - (E[X])^2 = 5 - 2^2 = 1.$$

Rules of Expectations and Variances

Proposition

- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

$\text{Var}(aX + b) = a^2 \, \text{Var}(X)$

Non-Random

① $\mathbb{E}[c] = c$

Non - Random

② $\mathbb{E}[a \cdot X] = a\,\mathbb{E}[X]$

③ $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$$\text{Var}(aX + b) \overset{\uparrow}{=} \mathbb{E}[(aX+b)^2] - (\mathbb{E}[aX+b])^2$$

Formula

$$= \mathbb{E}[a^2 X^2 + 2abX + b^2] - (a\,\mathbb{E}[X] + b)^2$$

$$= (a^2\,\mathbb{E}[X^2] + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2}) - (a^2 (\mathbb{E}[X])^2 + \cancel{2ab\,\mathbb{E}[X]} + \cancel{b^2})$$

$$= a^2 (\mathbb{E}[X^2] - (\mathbb{E}[X])^2) = a^2 \, \text{Var}(X).$$

Standard Deviation

$$Sd(X) = \sqrt{Var(X)}$$

$$\begin{aligned} Sd(aX+b) &= \sqrt{Var(aX+b)} \\ &= \sqrt{a^2\, Var(X)} \\ &= \sqrt{a^2} \cdot \sqrt{Var(X)} = \overset{|a|}{\overset{\|}{\sqrt{a^2}}} \cdot Sd(X) \end{aligned}$$

Q: $\quad E[X^2] \geq (E[X])^2$

$$\begin{aligned} Var(X) &= E[\underbrace{(X - E[X])^2}_{\geq 0}] \geq 0 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

In general,

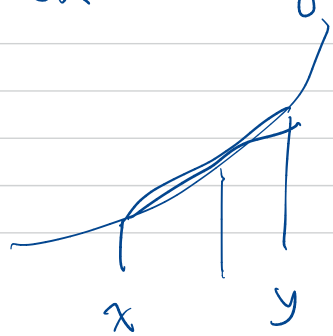
$$E[|X|^k] \geq |E[X]|^k \qquad k \geq 1$$

$$E[\phi(X)] \qquad \phi(E[X])$$

depends on the shape of ϕ

Jensen's Inequality. (Later).

ϕ : Convex iff $\quad \phi\left(\dfrac{x+y}{2}\right) \leq \dfrac{\phi(x)+\phi(y)}{2}$



Exercise

By 2:33.

(3.3-33) Let X be a Bernoulli RV with p.

Find the variance.

$$X = \begin{cases} 1 \\ 0 \end{cases} \qquad \mathbb{P}(X=1) = p$$

$$\mathbb{E}[X] = 0 \cdot p(0) + 1 \cdot p(1) = p$$

$$\mathbb{E}[X^2] = 0^2 \cdot p(0) + 1^2 \cdot p(1) = p$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p \cdot (1-p)$$

Q: For what p, smallest Var? $p=0, 1$

largest? $p = \frac{1}{2}$.

Q: $X \sim Geom(p)$

X = # of trials until 1^{st} Success

$$\mathbb{E}[X] = \frac{1}{p}$$

$$\mathbb{E}[X] = 1 \cdot p + 2 \cdot (1-p) \cdot p + 3 \cdot (1-p)^2 \cdot p + \cdots$$

$$(1-p)\, \mathbb{E}[X] = \qquad 1\,(1-p) \cdot p + 2\,(1-p)^2 \cdot p + \cdots$$

$$Var(X) = \underline{\mathbb{E}[X^2]} - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X^2] = 1^2 \cdot p + 2^2 \cdot (1-p) \cdot p + 3^2 \cdot (1-p)^2 \cdot p + \cdots$$

$$-\,(1-p)\, \mathbb{E}[X^2] = \qquad 1^2 (1-p) \cdot p + 2^2 (1-p)^2 \cdot p + \cdots$$

$$p\, \mathbb{E}[X^2] = 1 \cdot p + 3 \cdot (1-p) \cdot p + 5\,(1-p)^2\, p + 7\,(1-p)^3 \cdot p$$

$$(1-p) \cdot p\, \mathbb{E}[X^2] = \qquad 1\,(1-p)\,p + 3\,(1-p)^2\,p + \cdots$$

Section 4.
The Binomial Probability Distribution

Bernoulli Experiment = Exp. w/ Two outcomes
{S, F}

Binomial Experiments

Definition

An experiment consisting of a sequence of n trials is called a Binomial experiment if

1. Each trial can result in one of the same two possible outcomes (dichotomous trials), which we generically denote by success (S) and failure (F).
2. The trials are independent, so that the outcome on any particular trial does not influence the outcome on any other trial.
3. The probability of success $\mathbb{P}(S)$ is constant from trial to trial; we denote this probability by p.

Binomial Experiments

Example

In a box, there are 7 Green balls and 3 Red balls.

Consider an experiment with 3 trials: we draw 3 balls in a row with replacement.

Let A_i be the event that the i-th ball is Green for $i = 1, 2, 3$.

1. Find $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_3)$.
2. Are A_1, A_2 independent?
3. Is this experiment binomial?

1 $\mathbb{P}(A_1) = \frac{7}{10} = \mathbb{P}(A_2) = \mathbb{P}(A_3)$

2 $\mathbb{P}(A_1) \cdot \mathbb{P}(A_2) = \left(\frac{7}{10}\right)^2$

$\mathbb{P}(A_1 \cap A_2) = \frac{7 \times 7}{100}$ $\parallel$

Binomial Experiments

Example

In a box, there are 7 Green balls and 3 Red balls.

Consider an experiment with 3 trials: we draw 3 balls in a row without replacement.

Let A_i be the event that the i-th ball is Green for $i = 1, 2, 3$.

1. Find $\mathbb{P}(A_1), \mathbb{P}(A_2), \mathbb{P}(A_3)$.
2. Are A_1, A_2 independent?
3. Is this experiment binomial?

$$\mathbb{P}(A_1) = \frac{7}{10}.$$

$$\mathbb{P}(A_2) = \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1^c \cap A_2)$$

$$= \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 | A_1) + \mathbb{P}(A_1^c) \cdot \mathbb{P}(A_2 | A_1^c)$$

$$= \frac{7}{10} \cdot \frac{6}{9} + \frac{3}{10} \cdot \frac{7}{9} = \frac{42 + 21}{90} = \frac{63}{90}$$

$$= \frac{7}{10} = \mathbb{P}(A_3)$$

$$\mathbb{P}(A_1 \cap A_2) = \frac{7}{10} \cdot \frac{6}{9} \neq \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$$

$$\frac{7000}{10000} \cdot \frac{6999}{9999}$$

Binomial Experiments

Rule

Consider sampling without replacement from a dichotomous population of size N. If the sample size (number of trials) n is at most 5% of the population size, the experiment can be analyzed as though it were exactly a binomial experiment.

- Repeat Indep. $\underbrace{\text{Bernoulli}}_{\text{with } p}$ Exp $\underline{\underline{n}}$ times
- X = # of success in n trials

  $\sim \text{Bin}(n, p)$

The Binomial Random Variable and Distribution

Definition

The binomial random variable X associated with a binomial experiment consisting of n trials is defined as the number of Success among the n trials.

We denote by $X \sim \text{Bin}(n, p)$.

The Binomial Random Variable and Distribution

Example

Consider tossing a coin 5 times.

Assume that the probability of landing Heads is 0.4.

What is the probability having Heads 3 times?

X = # of Heads in 5 trials.

$\sim Bin(5, 0.4)$

$P(X = 3) = \binom{5}{3} (0.4) \cdot (0.4)(0.4)(0.6)(0.6)$

$= \binom{5}{3} (0.4)^3 (0.6)^2$

$\boxed{HHHTT}$

HTTHH

THTHH

⋮

The Binomial Random Variable and Distribution

The PMF and CDF of Binomial RV

Let $X \sim \mathrm{Bin}(n, p)$.

The PMF is

$$b(x; n, p) = \mathbb{P}(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

The CDF is

$$B(x; n, p) = \mathbb{P}(X \leq x) =$$

x many S, $(n-x)$ F

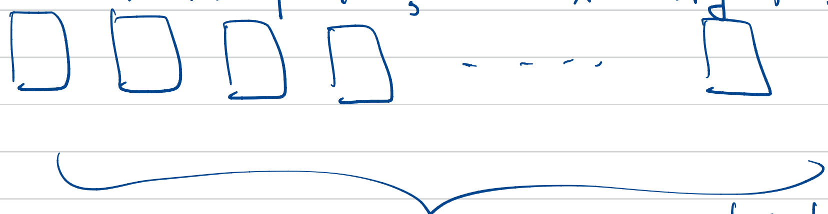
Recall

$X \sim Bin(n, p)$ $\quad X = \#$ of Success in n trials

PMF of X $= b(x; n, p) = p(x) = P(X = x)$

$$= \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0, 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

x many S's, $(n-x)$ many F.



n trials

CDF of X $= B(x; n, p) = F(x) = P(X \le x)$

$$= \sum_{k=0}^{\lfloor x \rfloor} p(k)$$

$\lfloor x \rfloor$ = the largest integer $\le x$

Note $\quad X = \#$ of success $\sim Bin(n, p)$

$Y = \#$ of F in the same exp.

$= n - X$

$Y \sim Bin(n, 1-p)$

15 copies

$\underline{\underline{N}}$ copies

The Binomial Random Variable and Distribution

Example

Suppose that 20% of all copies of a particular textbook fail a certain binding strength test.

Let X denote the number among 15 randomly selected copies that fail the test.

1. How is X distributed? under what assumptions?
2. Find the probability that at most 8 fail the test.
3. Find the probability that exactly 8 fail the test.

① $X \sim \text{Bin}(15, 0.2)$

② $P(X \leq 8) = \sum_{k=0}^{8} \binom{15}{k} (0.2)^k (0.8)^{15-k}$

③ $P(X = 8) = \binom{15}{8} (0.2)^8 \cdot (0.8)^{15-8}$.

The Binomial Random Variable and Distribution

c. $n = 15$

$\mathbb{P}(X \leq x)$

		p													
		0.01	**0.05**	**0.10**	**0.20**	**0.25**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.75**	**0.80**	**0.90**	**0.95**	**0.99**
	0	.860	.463	.206	.035	.013	.005	.000	.000	.000	.000	.000	.000	.000	.000	.000
	1	.990	.829	.549	.167	.080	.035	.005	.000	.000	.000	.000	.000	.000	.000	.000
	2	1.000	.964	.816	.398	.236	.127	.027	.004	.000	.000	.000	.000	.000	.000	.000
	3	1.000	.995	.944	.648	.461	.297	.091	.018	.002	.000	.000	.000	.000	.000	.000
	4	1.000	.999	.987	.836	.686	.515	.217	.059	.009	.001	.000	.000	.000	.000	.000
	5	1.000	1.000	.998	.939	.852	.722	.403	.151	.034	.004	.001	.000	.000	.000	.000
	6	1.000	1.000	1.000	.982	.943	.869	.610	.304	.095	.015	.004	.001	.000	.000	.000
x	7	1.000	1.000	1.000	.996	.983	.950	.787	.500	.213	.050	.017	.004	.000	.000	.000
	8	1.000	1.000	1.000	.999	.996	.985	.905	.696	.390	.131	.057	.018	.000	.000	.000
	9	1.000	1.000	1.000	1.000	.999	.996	.966	.849	.597	.278	.148	.061	.002	.000	.000
	10	1.000	1.000	1.000	1.000	1.000	.999	.991	.941	.783	.485	.314	.164	.013	.001	.000
	11	1.000	1.000	1.000	1.000	1.000	1.000	.998	.982	.909	.703	.539	.352	.056	.005	.000
	12	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	.973	.873	.764	.602	.184	.036	.000
	13	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.965	.920	.833	.451	.171	.010
	14	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.995	.987	.965	.794	.537	.140

$\mathbb{P}(X \leq 8)$

The Binomial Random Variable and Distribution

Example

An electronics manufacturer claims that at most 10% of its power supply units need service during the warranty period.

To investigate this claim, technicians at a testing laboratory purchase 20 units. Let p denote the probability that a power supply unit needs repair. The laboratory technicians must decide whether the data resulting from the experiment supports the claim that $p \leq .10$.

Let X denote the number among the 20 sampled that need repair.

Reject the claim that $p \leq .10$ in favor of the conclusion that $p > .10$ if $X \geq 5$.

Probability of wrong conclusion?

$X \sim \mathrm{Bin}(n, p)$

Possible values of X are : $0, 1, \cdots, n$

$p(0) + p(1) + \cdots + p(n) = 1$

$$\binom{n}{0} p^0 \cdot (1-p)^n + \binom{n}{1} p^1 \cdot (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} + \cdots$$
$$+ \cdots + \binom{n}{n} p^n (1-p)^0 = 1$$

$\boxed{\text{Thm}}$ $a, b \in \mathbb{R}$, $n \in \mathbb{N}$

$$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 \cdot b^{n-1} + \cdots + \binom{n}{n} a^n \cdot b^0$$

Binomial Thm!

In particular, if $a = p$, $b = 1-p$.

$$\binom{n}{0} p^0 \cdot (1-p)^n + \binom{n}{1} p^1 \cdot (1-p)^{n-1} + \binom{n}{2} p^2 (1-p)^{n-2} + \cdots$$
$$+ \cdots + \binom{n}{n} p^n (1-p)^0$$

$$= (p + (1-p))^n = 1^n = 1 .$$

$$\underline{\mathbb{E}}[X] = \sum_{k=0}^{n} k \cdot p(k) = \sum_{k=\cancel{0}\,1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$k \binom{n}{k} = k \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)! \big((n-1) - (k-1) \big)!}$$

$$= n \cdot \binom{n-1}{k-1}$$

$$E[X] = n \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k} (1-p)^{n-k}$$

$$p^{j+1} = p^{j} \cdot p^{1}$$

$$= np \boxed{\sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1-p)^{(n-1)-j}}$$

$$\begin{pmatrix} k-1 = j \\ k = j+1 \end{pmatrix}$$

$$= (p + (1-p))^{n-1} = 1$$

$$= n \cdot p .$$

$$Var(X) = \underbrace{E[X^2]}_{} - (\underbrace{E[X]}_{})^2$$

$$E[X(X-1)] = \sum_{k=0}^{n} \underbrace{k(k-1) \binom{n}{k}}_{} p^{k} (1-p)^{n-k}$$

$$= E[X^2 - X]$$

$$k(k-1) \frac{n!}{k! \, (n-k)!}$$

$$= E[X^2] - E[X]$$

$$Var(X) = n \cdot p (1-p),$$

The Binomial Random Variable and Distribution

Proposition

The mean and the variance of $X \sim \mathrm{Bin}(n, p)$ are

$$\mathbb{E}[X] = n \cdot p$$

$$\mathrm{Var}(X) = n \cdot p(1-p)$$

By 2:53.

Exercise

(3.4-52) Suppose that 30% of all students who have to buy a text for a particular course want a new copy (the successes!), whereas the other 70% want a used copy. Consider randomly selecting 25 purchasers.

1. What are the mean value and standard deviation of the number who want a new copy of the book?
2. What is the probability that the number who want new copies is more than two standard deviations away from the mean value? (X)
3. Suppose that new copies cost $100 and used copies cost $70. Assume the bookstore currently has 50 new copies and 50 used copies. What is the expected value of total revenue from the sale of the next 25 copies purchased?

X = # of New $\sim Bin(25, 0.3)$

Reasons.

1. $E[X] = 25 \cdot 0.3 = 7.5$

   $Var(X) = 25 \cdot 0.3 \cdot 0.7 = (7.5)(0.7) = 5.25$

   $\sigma_X = \sqrt{5.25} = 2.29$

2. $P(|X - E[X]| > 2\sigma)$

   $= 1 - P(\underbrace{E[X] - 2\sigma}_{2 \ldots} < X < \underbrace{E[X] + 2\sigma}_{12 \ldots}) =$

   $P(X = 0, 1, 2, 13, \ldots)$

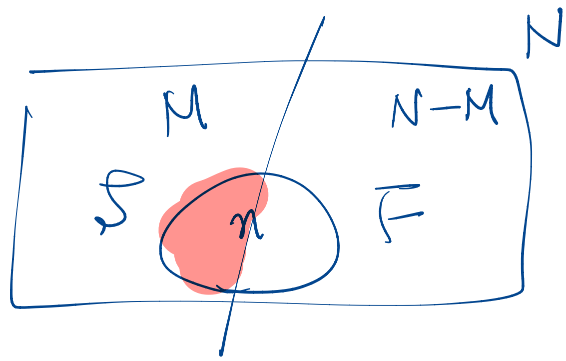
③ $R(X) = 100 \cdot X + 70 \cdot (25 - X)$

$= 70 \cdot 25 + 30X$

$E[R(X)] = E[70 \cdot 25 + 30X]$

$= 70 \cdot 25 + 30 \cdot \underbrace{E[X]}_{=7.5}$

Section 5. Hypergeometric and Negative Binomial Distributions



The Hypergeometric Distribution

Definition

1. The population or set to be sampled consists of N individuals, objects, or elements (a finite population).
2. Each individual can be characterized as a success (S) or a failure (F), and there are M successes in the population.
3. A sample of n individuals is selected without replacement in such a way that each subset of size n is equally likely to be chosen.

Total # S

Let X be the number of Success in the sample.

The random variable X is called hypergeometric and denoted by $X \sim \mathrm{HG}(n, M, N)$

Sample size

Population size.

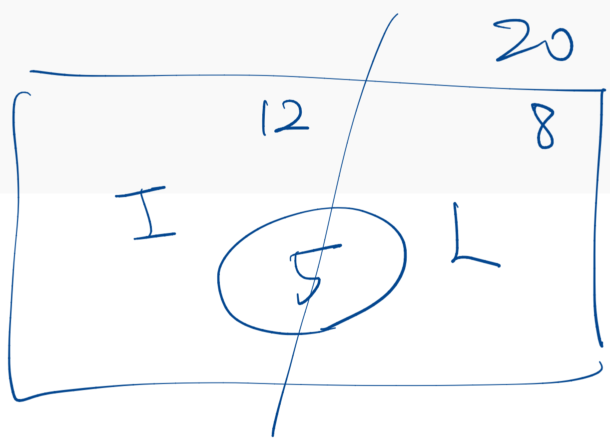
The Hypergeometric Distribution

Example

During a particular period a university's information technology office received 20 service orders for problems with printers, of which 8 were laser printers and 12 were inkjet models.

Suppose that the 5 are selected in a completely random fashion, so that any particular subset of size 5 has the same chance of being selected as does any other subset.

What then is the probability that exactly 3 of the selected service orders were for inkjet printers?



$$X = \# \text{ Inkjet } \sim HG(5, 12, 20)$$

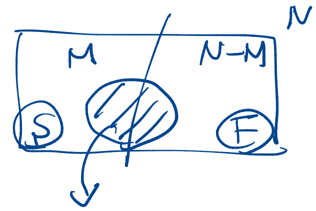
$$P(X=3) = \frac{\binom{12}{3}\cdot\binom{8}{2}}{\binom{20}{5}}$$

Population of size N

M many S / $(N-M)$ many F

Sample of size n without replacement

# of success among n chosen $\sim HG(n, M, N)$



The Hypergeometric Distribution

Proposition

Let $X \sim \mathrm{HG}(n, M, N)$.

The PMF is

$$p(x) = h(x; n, M, N) = \mathbb{P}(X = x) = \frac{\binom{M}{x} \cdot \binom{N-M}{n-x}}{\binom{N}{n}}$$

for x satisfying

for $\boxed{x = \; - \; - \; - \; - \; -}$?

$\max\{0, n-(N-M)\}$ $\quad$ $\min\{n, M\}$

Example $\quad n = 10, \; M = 7, \; N = 15$

15

7G	8R

$X = \#$ of G $\sim HG(10, 7, 15)$

$X = 2, \cdots, 7$

The Hypergeometric Distribution

Example

Five individuals from an animal population thought to be near extinction in a certain region have been caught, tagged, and released to mix into the population.

After they have had an opportunity to mix, a random sample of 10 of these animals is selected.

Let X be the number of tagged animals in the second sample.

If there are actually 25 animals of this type in the region, what is the probability that (a) $X = 2$ (b) $X \leq 2$?

$N = 25$, $M = 5$, $n = 10$

$X \sim HG(\underset{\sim}{10}, 5, 25)$

$$P(X=2) = \frac{\binom{5}{2}\binom{20}{8}}{\binom{25}{10}}$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

The Hypergeometric Distribution

Proposition

The mean and the variance of $X \sim \mathrm{HG}(n, M, N)$ are

$$\mathbb{E}[X] = n \cdot \frac{M}{N}$$

success probability

$$\mathrm{Var}(X) = \underbrace{\left(\frac{N-n}{N-1}\right)} \cdot n \cdot \frac{M}{N}\left(1 - \frac{M}{N}\right)$$

$Y \sim Bin(n, p)$

$Var(Y) = n \cdot p \cdot (1-p)$

Different from Binomial,

Repeat Independent Bernoulli Experiments

Stop when r many success.

$X = \#$ of ~~trials~~ Failure $\sim NB(r, p)$

$\#$ of S required — success prob.

In particular, $r=1$ $NB(1,p)^{+1} = Geom(p)$

The Negative Binomial Distribution

Definition

1. The experiment consists of a sequence of independent trials.
2. Each trial can result in either a success (S) or a failure (F).
3. The probability of success is constant from trial to trial.
4. The experiment continues (trials are performed) until a total of r successes have been observed, where r is a specified positive integer.

Let X be the number of failures that precede the r-th success.

The random variable X is called Negative Binomial and denoted by $X \sim \text{NegBin}(r, p)$.

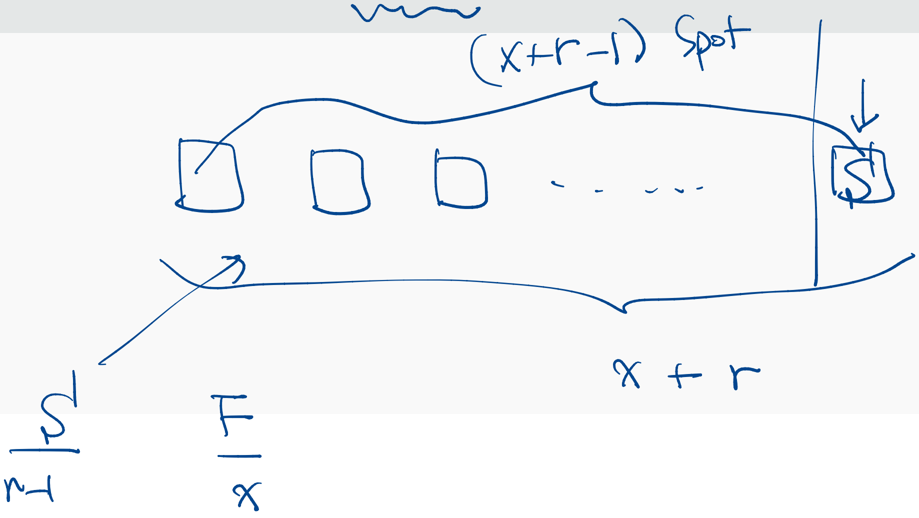
The Negative Binomial Distribution

X = # of F until r^{th} success

Proposition

The PMF of $X \sim \text{NegBin}(r, p)$ is

$$nb(x; r, p) = \mathbb{P}(X = x) =$$



$$P(X=x) = \binom{x+r-1}{r-1} \cdot p^r \cdot (1-p)^x$$

for $x = 0, 1, \ldots$

The Negative Binomial Distribution

Example

A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. Let

$$p = \mathbb{P}(\text{a randomly selected couple agrees to participate}).$$

If $p = .2$, what is the probability that 15 couples must be asked before 5 are found who agree to participate?

Y = # of couples asked before 5 couples agrees

$X = Y - 5 \sim NB(5, 0.2)$

$$P(Y = 15) = P(X = 10) = \binom{14}{4} p^5 \cdot (1-p)^{10}$$

$$D \; \underbrace{- \; - \; - \; - \; - \; S}_{15}$$

Binomial Thm

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$X \sim NB(r, p) \qquad X = 0, 1, 2, \dots$

$1 = P(X=0) + P(X=1) + \dots$

$= \binom{0+r-1}{r-1} p^r \cdot (1-p)^0 + \binom{1+r-1}{r-1} p^r \cdot (1-p)^1 + \binom{2+r-1}{r-1} p^r \cdot (1-p)^2 + \dots$

The Negative Binomial Distribution

Proposition

The mean and the variance of $X \sim \mathrm{NegBin}(r, p)$ are

$$\mathbb{E}[X] = r \cdot \left(\frac{1}{p} - 1\right)$$

$$\mathrm{Var}(X) = r \cdot \frac{(1-p)}{p^2}$$

$X \sim NB(4, p)$ $\qquad$ $Y = \#$ of trials $= X + 4$

$$\overbrace{F \quad F \quad F \quad S}^{Y_1} \quad \overbrace{F \quad S}^{Y_2} \quad \overbrace{F \quad F \quad S}^{Y_3} \quad \overbrace{F \quad F \quad F \quad F \quad S}^{Y_4}$$

$$Y = \underset{\text{Geom}(p)}{\underline{Y_1}} + \underset{\text{Geom}(p)}{Y_2} + Y_3 + Y_4$$

Indep

$$E[Y] = \underline{E[Y_1]} + \underline{E[Y_2]} + \underline{E[Y_3]} + \underline{E[Y_4]}$$

$$= 4 \cdot \frac{1}{p}.$$

$$E[X] = 4 \cdot \frac{1}{p} - 4 = 4 \cdot \left(\frac{1}{p} - 1\right)$$

By 2:54

Exercise

(3.5-71) A geologist has collected 10 specimens of basaltic rock and 10 specimens of granite. The geologist instructs a laboratory assistant to randomly select 15 of the specimens for analysis.

1. What is the PMF of the number of granite specimens selected for analysis?
2. What is the probability that all specimens of one of the two types of rock are selected for analysis?
3. What is the probability that the number of granite specimens selected for analysis is within 1 standard deviation of its mean value?

X = # of granite $\sim HG(15, 10, 20)$

① For $x = 5, \cdots, 10$,

$$p(x) = P(X = x) = \frac{\binom{10}{x}\binom{10}{15-x}}{\binom{20}{15}}$$

② $$P(X=10) + P(X=5) = \frac{\binom{10}{10}\cdot\binom{10}{5} + \binom{10}{5}\binom{10}{10}}{\binom{20}{15}}$$

③ $$P(|X - E[X]| < \sigma_X)$$

$$= P(7.5 - 0.993 < X < 7.5 + 0.993) = P(X = 7, 8)$$

$$E[X] = 15 \cdot \frac{1}{2} = 7.5$$

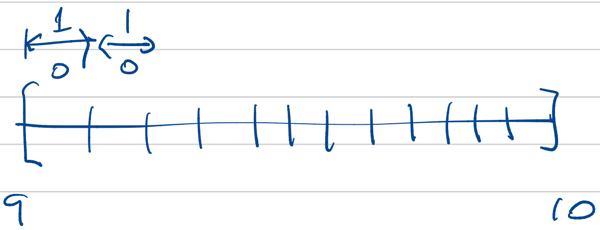
$$Var(X) = 15 \cdot \frac{1}{2} \cdot \left(1 - \frac{1}{2}\right) \cdot \frac{N-n}{N-1} = 0.987$$

$$\frac{N-n}{N-1} = \frac{5}{19}$$

$$\sigma = 0.993$$

Section 6.
The Poisson Probability Distribution

Poisson RV



$n = 60$

X = # of customers between 9 & 10.

5 customers in average

$$\underbrace{[\quad]\ [\quad]\ [\quad]\ \cdots\ [\quad]}$$

$$X \approx Bin\left(n, \frac{5}{n}\right) \xrightarrow{n\to\infty} \text{Poisson}$$

$$\uparrow \quad n \cdot p = 5$$

PMF $\xrightarrow{n\to\infty}$

$$e = 2.7\cdots = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =$$

Possible values for $X = 0, 1, 2, \cdots$

Poisson Random Variables

of Incoming customers for 1 hr.

Definition

A discrete random variable X is said to have a Poisson distribution with parameter $\mu > 0$ if the PMF of X is

$$p(x;\mu) = \mathbb{P}(X = \cancel{x}\ k) = \frac{e^{-\mu}\mu^k}{k!}$$

Average # of Customers.

We denote by $X \sim \text{Pois}(\mu)$.

1. The parameter μ stands for the mean of X
2. Recall that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

3. $k! = k(k-1)\cdots 2\cdot 1$

If $X \sim \text{Pois}(\mu)$

$$P(X=0) + P(X=1) + P(X=2) + \cdots \quad \left(\overset{\text{Should be}}{=} 1\right)$$

$$= \frac{e^{-\mu}\cdot\mu^0}{0!} + \frac{e^{-\mu}\mu^1}{1!} + \frac{e^{-\mu}\mu^2}{2!} + \cdots$$

$$= e^{-\mu}\underbrace{\boxed{\sum_{k=0}^{\infty}\frac{\mu^k}{k!}}}_{=e^{\mu}} = e^{-\mu}\cdot e^{+\mu} = e^{-\mu+\mu} = e^0 = 1$$

Poisson Random Variables

Proposition

The mean and the variance of $X \sim \text{Pois}(\mu)$ are

$$\mathbb{E}[X] = \mu$$
$$\text{Var}(X) = \mu$$

Why?

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k \cdot \mathbb{P}(X = k)$$

$$= \sum_{k=\cancel{0}\,1}^{\infty} \cancel{k} \cdot e^{-\mu} \cdot \frac{\mu^{k-1} \cdot \mu}{\cancel{k!}\,(k-1)!}$$

$$= e^{-\mu} \underbrace{\left(\sum_{j=0}^{\infty} \frac{\mu^{j}}{j!} \right)}_{= e^{\mu}} \cdot \mu = \mu$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \underbrace{\mathbb{E}[X(X-1)]}_{= \sum k(k-1) \cdot \mathbb{P}(X=k)} + \underline{\mathbb{E}[X]} - (\mathbb{E}[X])^2$$

$$= \mu^2 + \mu - \mu^2 = \mu$$

Poisson Random Variables

Example

Let X denote the number of creatures of a particular type captured in a trap during a given time period.

Suppose that X has a Poisson distribution with $\mu = 4.5$, so on average traps will contain 4.5 creatures.

What is the probability that a trap contains exactly five creatures?

What are the expectation and the variance?

$$X \sim \text{Pois}(4.5)$$

$$P(X = 5) = \frac{e^{-4.5} (4.5)^5}{5!}$$

$$E[X] = 4.5 = \text{Var}(X).$$

- Discretize Time Intervals
- Each time interval: No customer, $1-p$; 1 customer, p

Indep.

$\Rightarrow Y_n = \text{Bin}(n, p) \approx \text{Pois}(\mu)$, $p \leftarrow \frac{\mu}{n}$ ← constant

$n \cdot p = \mu$

$p = \frac{\mu}{n}$

The Poisson Distribution as a Limit

$$P(Y_n = k) = \binom{n}{k} \cdot \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} \xrightarrow[n\to\infty]{} \frac{e^{-\mu}\mu^k}{k!}$$

Proposition

Suppose that in the binomial PMF $b(x; n, p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value μ. Then

$$b(x; n, p) \to p(x; \mu).$$

If $n \to \infty$

$p \to 0$

$np \to \mu$

Then, $\text{Bin}(n, p) \approx \text{Pois}(\mu)$

The Poisson Distribution as a Limit

According to this proposition, in any binomial experiment in which n is large and p is small,

$$b(x; n, p) \approx p(x; \mu),$$

where $\mu = np$.

As a rule of thumb, this approximation can safely be applied if $n > 50$ and $np < 5$.

Example

If a publisher of nontechnical books takes great pains to ensure that its books are free of typographical errors, so that the probability of any given page containing at least one such error is .005 and errors are independent from page to page,

what is the probability that one of its 400-page novels will contain exactly one page with errors? At most three pages with errors?

400 pages

Each page $\begin{cases} \text{errors}, & p = 0.005 \\ \text{Not}, & 1-p \end{cases}$

$X = \#$ of pages w/ errors

$\sim \text{Bin}(400, 0.005)$ $\qquad np = 2 = \mu$

$\approx \text{Pois}(2)$

$P(X = 1) = \binom{400}{1}(0.005)^1(0.995)^{399}$

$$\approx \frac{e^{-2} \cdot 2^{1}}{1!}$$

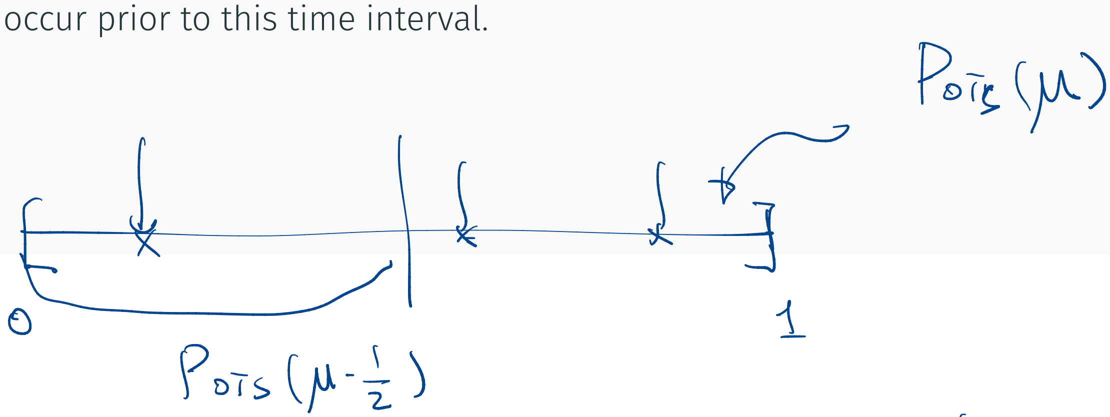
$$P(X \leq 3) \approx e^{-2}\left(1 + \frac{2^{1}}{1!} + \frac{2^{2}}{2!} + \frac{2^{3}}{3!}\right)$$

$$\frac{o(\Delta t)}{\Delta t} \to 0 \quad \text{as} \quad \Delta t \to 0$$

The Poisson Process

Events of interest might be visits to a particular website, pulses of some sort recorded by a counter, email messages sent to a particular address, accidents in an industrial facility, or cosmic ray showers observed by astronomers at a particular observatory. We make the following assumptions about the way in which the events of interest occur:

1. There exists a parameter $\alpha > 0$ such that for any short time interval of length Δt, the probability that exactly one event occurs is $\alpha \Delta t + o(\Delta t)$.
2. The probability of more than one event occurring during Δt is $o(\Delta t)$.
3. The number of events occurring during the time interval Δt is independent of the number that occur prior to this time interval.



Customers in $[0, t] \sim \text{Pois}(\mu t)$

The Poisson Process

Proposition

The number of events during a time interval of length t is a Poisson RV with parameter $\mu = \alpha t$.

The Poisson Process

Example

Suppose pulses arrive at a counter at an average rate of six per minute, so that $\alpha = 6$.

Find the probability that in a .5-min interval at least one pulse is received.

X = # of pulses in 0.5 min

$\sim \text{Pois}(\underline{\ 3\ })$

$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X \leq 0) = 1 - e^{-3}$

Exercise

(3.6-86) The number of people arriving for treatment at an emergency room can be modeled by a Poisson process with a rate parameter of five per hour.

1. What is the probability that exactly four arrivals occur during a particular hour?
2. What is the probability that at least four people arrive during a particular hour?
3. How many people do you expect to arrive during a 45-min period?

X_t = # of people in $[0, t]$

$\sim \text{Pois}(5t)$

① $\mathbb{P}(X_1 = 4) = \dfrac{e^{-5} \cdot 5^4}{4!}$

③ $\mathbb{E}[X_{\frac{3}{4}}] = 5 \cdot \frac{3}{4}$