Practice Problems for Final Exam

Part 1

1. The following numbers x_i, $i = 1, ..., 17$, represent a sample of size $n = 17$ from a given population.

3.0273	2.8598	3.2407	3.3058	3.0626	3.1718
2.5795	2.4375	2.4639	2.8931	2.8598	3.4566
3.1399	3.2174	2.9072	2.6108	3.2846	

   (a) Compute the sample median and fourth spread.

   (b) Knowing that

$$\sum_{i=1}^{17} x_i = 50.5182, \qquad \sum_{i=1}^{17} x_i^2 = 151.6426,$$

   compute the sample mean and the variance.

   (c) Draw a box plot of the data.

2. A box in a certain supply room contains four 40W light bulbs, five 60W bulbs, and six 75W bulbs.

   (a) Suppose that three bulbs are randomly selected. What is the probability that exactly two of the selected bulbs are rated 75W?

   (b) Suppose now that bulbs are to be selected one by one until a 75W bulb is found. What is the probability that it is necessary to examine at least six bulbs?

3. One percent of all individuals in a certain population are carriers of a particular disease. A diagnostic test for this disease has a 90% detection rate for carriers and a 5% detection rate for non-carriers. Suppose that the test is applied independently to two different blood samples from the same randomly selected individual.

   (a) What is the probability that both tests are positive?

   (b) Given that both tests are positive, what is the conditional probability that the selected individual is a carrier?

4. A toll bridge charges \$1.00 for passenger cars and \$2.50 for other vehicles. Suppose that during daytime hours, 60% of all vehicles are passenger cars. Suppose that 25 vehicles cross the bridge during a particular day.

   (a) Let X be the number of passenger cars among 25 vehicles. How is X distributed?

   (b) What is the expected number of passenger cars among 25 vehicles, that is, $\mathbb{E}[X]$?

   (c) Find the expression of the toll revenue from the 25 vehicles $h(X)$ in terms of X. Compute the expected revenue $\mathbb{E}[h(X)]$.

5. Two six-sided dice are tossed independently. Let X be the maximum of the two outcomes. (For example, if two outcomes are 1 and 5, then $X = 5$.)

   (a) Let $p(x)$ be the PMF of X. Find $p(3)$ and $p(4)$.

   (b) Let $F(x)$ be the CDF of X. Find $F(0)$ and $F(3.2)$.

6. In Atlanta there are 2,000,000 families. Among them 40,000 do not report correctly their incomes. The IRS selects a sample of 200 families and controls their tax returns. Let X be the number of incorrect reports among these 200 families.

   (a) What is the probability distribution of X? Write a formula for the probability that $X = 4$.

   (b) Use a binomial approximation to compute the average and variance of X. Justify the approximation.

7. In a bowl, there are 10 red balls, 20 green balls, and 30 blue balls. You randomly chose 9 out of them without replacement.

(a) Find the probability that 4 of the 9 chosen balls are red.

(b) Find the probability that 6 of the 9 chosen balls are red or green.

(c) Find the probability that 3 of the 9 chosen balls are red, 3 are blue, and 3 are green.

8. A customer purchases 10 bulbs from a store. We know that each bulb has the probability $p = 0.99$ to be working independently from the others.

   (a) What is the probability that all bulbs are working?

   (b) What is the probability that exactly 4 out of the 10 bulbs are working?

   (c) What is the expected value of the number of working bulbs? and its variance?

Part 2

1. Let X and Y be discrete random variables with joint PMF $p(x, y)$ given by

$$p(0,0) = p(0,1) = p(1,1) = p(1,2) = \frac{1}{4}.$$

   (a) Find the conditional expectation $\mathbb{E}[X|Y = 1]$.

   (b) Find the covariance $\mathrm{Cov}(X, Y)$.

   (c) Are they independent?

2. Let X, Y be continuous random variables with joint PDF

$$f(x,y) = \begin{cases} Cxy, & 0 \le x \le 1, 0 \le y \le 1, y \ge x \\ 0 & \text{otherwise} \end{cases}$$

   for some constant $C > 0$.

   (a) Find the constant C.

   (b) Find the marginal PDF of Y.

   (c) Find the conditional probability $\mathbb{P}(Y \ge 3/4|X = 1/2)$.

3. Let $\Phi(x)$ be the CDF of the standard normal distribution.

   (a) Let $X \sim N(3, 16)$. Write down $\mathbb{P}(1 \le X < 4)$ in terms of $\Phi(x)$ for $x \ge 0$. Then use the corresponding table to compute the probability.

   (b) Let X, Y be independent normal random variables with common distribution $N(3, 16)$. Use the corresponding table to compute the probability $\mathbb{P}(1 \le X < 2, 4 < Y \le 5)$.

4. Let $X \sim \mathrm{Exp}(\lambda)$ and $h(x)$ be a function defined by

$$h(x) = \begin{cases} 1, & x \le 1 \\ 2, & x > 1. \end{cases}$$

   Compute the expectation $\mathbb{E}[h(X)]$.

5. Roll two 4-sided dice. Let X be the outcome of the first die and Y be the sum of two outcomes.

   (a) Find the conditional expectation $\mathbb{E}[X|Y = 4]$.

   (b) Find the covariance $\mathrm{Cov}(X, Y)$.

   (c) Are they independent?

6. Let X be an exponential RV with parameter $\lambda = 2$ and Y be a uniform RV between 0 and 1. Assume that X and Y are independent.

   (a) Compute the joint probability distribution function $f(x, y)$ of X and Y.

   (b) Compute the probability that $X > 3$ and $Y > 0.5$.

   (c) Compute the probability that X is larger than Y.

7. A passenger can take two different bus lines to get to work. Both lines stop at the same bus stop. Let T_1 be the RV that gives the time of arrival of the bus of line 1 and T_2 the RV giving the time of arrival of the bus of line 2. You know that T_1 is exponential with parameter 10 and T_2 is exponential with parameter 20 when time is measured in minutes. Let T be the random variable that gives the time our passenger will wait for a bus, i.e., $T = \min(T_1, T_2)$ the smallest between T_1 and T_2.
   (a) Compute $\mathbb{P}(T > t)$.
   (b) What is the PDF of T?
   (c) What is the average time the passenger will wait at the bus stop?
8. Let $X \sim N(0, 4)$ and $W = X^2$. Find the PDF of W.
9. Let X and Y have the joint PDF
$$f(x, y) = \frac{4}{3}, \qquad 0 < x < 1, \quad x^3 < y < 1$$
   and 0 otherwise. Find $\mathbb{P}(X > Y)$.
10. A certain type of electrical motors is defective with probability 1/100. Pick 1000 motors and let X be the number of defective ones among these 1000 motors. Using a normal approximation, (with/without) mid-point correction, write down an expression for the probability that among the 1000 motors 13 or less are defective.

Part 3

1. Let X_1 and X_2 be two independent normal standard RVs. Let
$$Y_1 = X_1 + X_2$$
$$Y_2 = X_1 - X_2$$
   (a) Compute the expected value and variance of Y_1 and Y_2.
   (b) Compute $\text{Cov}(Y_1, Y_2)$ and $\text{Corr}(Y_1, Y_2)$.
2. Let Z be a discrete random variable with $\mathbb{P}(Z = 1) = \mathbb{P}(Z = -1) = \frac{1}{2}$. Let $X \sim N(0, 1)$ and define $Y := XZ$.
   (a) What is the distribution of Y?
   (b) Find $\text{Cov}(X, Y)$.
   (c) Are X and Y independent?
3. The time that it takes a randomly selected rat of a certain subspecies to find its way through a maze is a normally distributed RV with $\mu = 1.5$ min and $\sigma = .35$ min. Suppose that five rats are selected. Let $X_1, \cdots, X_5$ denote their times in the maze. Assuming the X_i's to be a random sample from this normal distribution, what is the probability that the total time is between 6 and 8 min?
4. A certain consumer organization customarily reports the number of major defects for each new automobile that it tests. Suppose the number of such defects for a certain model is a random variable with mean value 3.2 and standard deviation 2.4. Among 100 randomly selected cars of this model, how likely is it that the sample average number of major defects exceeds 4?
5. A binary communication channel transmits a sequence of "bits" (0s and 1s). Suppose that for any particular bit transmitted, there is a 10% chance of a transmission error (a 0 becoming a 1 or a 1 becoming a 0). Assume that bit errors occur independently of one another.
   (a) Consider transmitting 1000 bits. What is the approximate probability that at most 125 transmission errors occur?
   (b) Suppose the same 1000-bit message is sent two different times independently of one another. What is the approximate probability that the number of errors in the first transmission is within 50 of the number of errors in the second?
6. A certain automobile manufacturer equips a particular model with either a six-cylinder engine or a four-cylinder engine. Let X_1 and X_2 be fuel efficiencies for independently and randomly selected six-cylinder and four-cylinder cars, respectively, with
$$\mu_1 = 22, \qquad \mu_2 = 26, \qquad \sigma_1 = 1.2, \qquad , \sigma_2 = 1.5.$$
   Find $\mathbb{E}[X_1 - X_2]$ and $\text{Var}(X_1 - X_2)$.

7. Manufacture of a certain component requires three different machining operations. Machining time for each operation has a normal distribution, and the three times are independent of one another. The mean values are 15, 30, and 20 min, respectively, and the standard deviations are 1, 2, and 1.5 min, respectively. What is the probability that it takes at most 1 hour of machining time to produce a randomly selected component?

8. Let $X_1, \cdots, X_n$ be a random sample from a uniform distribution on $[0, \theta]$. Consider

$$\widehat{\theta}_1 = \frac{n+1}{n} \max X_i$$
$$\widehat{\theta}_2 = 2\overline{X}.$$

Are they unbiased? Find the variances of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

9. In a random sample of 80 components of a certain type, 12 are found to be defective. Give a point estimate of the proportion of all such components that are not defective.

10. Let $X_1, \cdots, X_n$ represent a random sample of service times of n customers at a certain facility, where the underlying distribution is assumed exponential with parameter λ. Find the moment estimator for λ.

11. (a) Let $X_1, \cdots, X_n$ be a random sample from Bernoulli distribution. Find the Likelihood function and the MLE for p.

    (b) Let $X_1, \cdots, X_n$ be a random sample from exponential distribution. Find the Likelihood function and the MLE for λ.

    (c) Let $X_1, \cdots, X_n$ be a random sample from normal distribution. Find the Likelihood function and the MLEs for μ, σ^2.

12. Two different computer systems are monitored for a total of n weeks. Let X_i denote the number of breakdowns of the first system during the i-th week, and suppose the X_i's are independent and drawn from a Poisson distribution with parameter μ_1. Similarly, let Y_i denote the number of breakdowns of the second system during the i-th week, and assume independence with each Y_i Poisson with parameter μ_2. Derive the MLE's of μ_1, μ_2, and $\mu_1 - \mu_2$