Midterm 3 Lecture Review Activity, Math 1554

1. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If S is a two-dimensional subspace of $\mathbb{R}^{50}$, then the dimension of $S^\perp$ is 48.	⊗	○
b) An eigenspace is a subspace spanned by a single eigenvector.	○	⊗
c) The $n \times n$ zero matrix can be diagonalized.	⊗	○
d) A least-squares line that best fits the data points $(0, y_1), (1, y_2), (2, y_3)$ is unique for any values y_1, y_2, y_3.	⊗	○

$\dim(\underbrace{S}_{2}) + \dim(S^\perp) = \dim(\underbrace{\mathbb{R}^{50}}_{50})$

$= \mathrm{Nul}(A - \lambda I)$

2. If possible, give an example of the following.

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2.1) A matrix, A, that is in echelon form, and $\dim\left((\mathrm{Row}A)^\perp\right) = 2$, $\dim\left((\mathrm{Col}\,A)^\perp\right) = 1$

$\mathrm{Row}(A)^\perp = \mathrm{Col}(A^T)^\perp = \mathrm{Nul}((A^T)^T) = \mathrm{Nul}(A)$

$\dim(\mathrm{Row}(A)^\perp) = \dim(\mathrm{Nul}(A)) = \#$ of free variables

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\det(A) = 0$

not invertible

2.2) A singular 2×2 matrix whose eigenspace corresponding to eigenvalue $\lambda = 2$ is the line $x_1 = 2x_2$. The other eigenspace of the matrix is the x_2 axis.

$=A$

$\lambda_1 = 2 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \lambda_2 = 0 \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}^{-1}$

$A = P \cdot D \cdot P^{-1}$

rank thm

$\dim(\mathrm{Col}(A)) \longrightarrow \ \Vert \ \ \dim(\mathrm{Row}(A))$

$0 = \det(A) = \det(P \cdot D \cdot P^{-1}) = \cancel{\det P} \cdot \det D \cdot \cancel{\det(P^{-1})} = \det(D) = \det\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 \cdot \lambda_2$

2.3) A subspace S, of $\mathbb{R}^4$, that satisfies $\dim(S) = \dim(S^\perp) = 3$.

N.P.

$\mathbb{R}^n \xrightarrow{A \in \mathbb{R}^{m \times n}} \mathbb{R}^m$

$\mathrm{Col}(A) \in \mathbb{R}^m$

$\mathrm{Row}(A) \in \mathbb{R}^n$

2.4) A 2×3 matrix, A, that is in RREF. $\underbrace{(\mathrm{Row}\,A)^\perp}_{= \mathrm{Nul}(A)}$ is spanned by $A\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$.

$\dim(\mathrm{Col}(A)^\perp) = m - \dim(\mathrm{Col}(A))$

$\dim(\mathrm{Row}(A)^\perp) = n - \dim(\mathrm{row}(A))$

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A\vec{x} = \vec{b} \qquad A \in \mathbb{R}^{m\times n}$$

Least squares solution is unique

$\Leftrightarrow$ Columns of A are linearly indep.

$\Leftrightarrow$ $A^T A$ is invertible

- $(0, y_1)$ $(1, y_2)$ $(2, y_3)$
- $y = \beta_0 + \beta_1 x$

$$\Rightarrow \begin{cases} y_1 = \beta_0 + \beta_1 \cdot 0 \\ y_2 = \beta_0 + \beta_1 \cdot 1 \\ y_3 = \underbrace{\beta_0 + \beta_1 \cdot 2} \end{cases}$$

$\Downarrow$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

unique least squares line. $\Leftarrow$ has lin. indep columns. (not scalar multiple of each other)

3. Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. For the situations that are possible give an example.

   3.1) A is $n \times n$, $A\vec{x} = A\vec{y}$ for a particular $\vec{x} \neq \vec{y}$, $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, and $\dim((\text{Row } A)^{\perp}) \neq 0$.

   $\Updownarrow$

   $A(x-y) = 0$

   $\Updownarrow$

   $x - y \in \text{Nul}(A)$

   possible **impossible**

   $\underbrace{\text{Nul}(A) \neq 0}$

   $\exists \; \underline{z} \neq 0 \in \text{Nul}(A)$

   3.2) A is $n \times n$, $\lambda \in \mathbb{R}$ is an eigenvalue of A, and $\dim((\text{Col}(A - \lambda I))^{\perp}) = 0$.

   possible **impossible**

   $\text{Nul}(A) = \{0\}$

   $\dim(\text{Nul}(A)) = 0$

   $\Updownarrow$

   A is 1-1

   $\Updownarrow$

   $Ax = Ay$ implies $x = y$.

   3.3) $\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\vec{u}}\vec{v}$, $\vec{v} \neq \vec{u}$, and $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$.

   possible **impossible**

4. Consider the matrix A.

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Construct a basis for the following subspaces and state the dimension of each space.

   4.1) $(\text{Row} A)^{\perp}$ $= \text{Nul}(A)$ $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

   4.2) Col A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

   4.3) $(\text{Col} A)^{\perp}$ $= \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

A matrix $U \in \mathbb{R}^{n \times n}$ is <u>orthogonal</u> if columns are orthonormal.

U is orthogonal $\Leftrightarrow$ $U^T \cdot U = \begin{pmatrix} \overset{=1}{u_1 \cdot u_1} & \overset{=0}{u_1 \cdot u_2} & \cdots & \overset{=0}{u_1 \cdot u_n} \\ \underset{=0}{u_2 \cdot u_1} & \overset{=1}{u_2 \cdot u_2} & \cdots & u_2 \cdot u_n \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$

$U = [u_1, u_2, \cdots, u_n]$

$= I$

$1 = \det(U^T \cdot U) = \underset{\overset{\shortparallel}{\det(U)}}{\det(U^T)} \cdot \det(U) = \det(U)^2$

$\Rightarrow \det(U) = 1, -1$

<u>Ex</u> $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ orthonormal

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: orthogonal matrix

$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ $\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$