MATH 461 MIDTERM 2 PRACTICE

1. Exam info

(1) **Time and Place**: April 13, in class (1–1:50pm).
(2) **Coverage**: Sections 5.1–7, 6.1–7 (up to material covered in Apr 6 class)
(3) No book or notes are allowed. Using calculator is okay.
(4) **Types of problems** may include:
   (a) Stating definitions
   (b) Write the statements and give proofs for theorems (or principles) covered in class
   (c) True and False
   (d) Giving examples
   (e) Computations for concrete examples

2. Review

2.1. Continuous Random Variables.

(1) Definition of continuous RVs
(2) Relation between Density and Distribution
(3) Expectation, Variance, and Expectation of a function of RV.
(4) Uniform $X \sim \mathrm{Unif}(a, b)$
   (a) $\mathbb{E}[X] = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.
(5) Normal $X \sim N(\mu, \sigma^2)$
   (a) $\mathbb{E}[X] = \mu$, $\mathrm{Var}(X) = \sigma^2$.
   (b) $X = \sigma Z + \mu$ for $Z \sim N(0, 1)$.
   (c) $\Phi(x) = \mathbb{P}(X \leq x)$, $\Phi(x) = 1 - \Phi(-x)$.
   (d) Normal approximation to binomial.
(6) Exponential $X \sim \mathrm{Exp}(\lambda)$
   (a) $\mathbb{E}[X] = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.
   (b) Memoryless property
(7) Gamma $X \sim \mathrm{Gamma}(\alpha, \lambda)$
   (a) $\mathbb{E}[X] = \frac{\alpha}{\lambda}$, $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$.
   (b) Relation to Poisson

2.2. Jointly Distributed Random Variables.

(1) Joint distribution

(a) Definitions of joint distribution, joint PMF, joint density.

(b) Definition of jointly continuous RVs.

(c) Marginal distribution, marginal densities.

(d) Definition of Independent RVs.

(2) Independent sum: Let X, Y be independent jointly continuous RVs.

(a) $F_{X+Y}(t) = \int F_X(t-y) f_Y(y)\, dy$.

(b) $f_{X+Y}(t) = \int f_X(t-y) f_Y(y)\, dy$.

(3) Conditional distribution

(a) Conditional PMF: If $p_Y(y) \neq 0$, then $p_{X|Y}(x|y) = \mathbb{P}(X = x | Y = y) = \frac{p(x,y)}{p_Y(y)}$.

(b) Conditional PDF: If $f_Y(y) \neq 0$, then $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$.

3. Practice problems

Q1) The random variable X ha sthe density

$$f(x) = \begin{cases} ax + bx^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathbb{E}[X] = 0.6$, find $\mathbb{P}(X < \frac{1}{2})$ and $\mathrm{Var}(X)$.

Q2) Your company must make a sealed bid for a construction project. If you succeed in winning the contract (by having the lowest bid), then you plan to pay another firm \$100,000 to do the work. If you believe that the minimum bid (in thousands of dollars) of the other participating companies can be modeled as the value of a random variable that is uniformly distributed on $(70, 140)$, how much should you bid to maximize your expected profit?

Q3) The life of a certain type of automobile tire is normally distributed with mean 34,000 miles and standard deviation 4000 miles.

(a) What is the probability that such a tire lasts more than 40,000 miles?

(b) What is the probability that it lasts between 30,000 and 35,000 miles?

(c) Given that it has survived 30,000 miles, what is the conditional probability that the tire survives another 10,000 miles?

Q4) At a certain bank, the amount of time that a customer spends being served by a teller is an exponential random variable with mean 5 minutes. If there is a customer in service when you enter the bank, what is the probability that he or she will still be with the teller after an additional 4 minutes?

Q5) Let X be a random variable with density

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Find the density of $1/X$.

Q6) The joint probability mass function of the random variables X, Y, Z is

$$p(1,2,3) = p(2,1,1) = p(2,2,1) = p(2,3,2) = \frac{1}{4}.$$

Find $\mathbb{E}[XYZ]$ and $\mathbb{E}[XY + YZ + ZX]$.

Q7) Let X and Y be continuous random variables with joint density function

$$f(x,y) = \begin{cases} \frac{x}{5} + cy, & 0 < x < 1, 1 < y < 5, \\ 0, & \text{otherwise.} \end{cases}$$

Find c and $\mathbb{P}(X + Y > 3)$. Are X, Y independent?

Q8) Consider two components and three types of shocks. A type 1 shock causes component 1 to fail, a type 2 shock causes component 2 to fail, and a type 3 shock causes both components 1 and 2 to fail. The times until shocks 1, 2, and 3 occur are independent exponential random variables with respective rates λ_1, λ_2, and λ_3. Let X_i denote the time at which component i fails, $i = 1, 2$. Find $\mathbb{P}(X_1 > s, X_2 > t)$.

Q9) A model proposed for NBA basketball supposes that when two teams with roughly the same record play each other, the number of points scored in a quarter by the home team minus the number scored by the visiting team is approximately a normal random variable with mean 1.5 and variance 6. In addition, the model supposes that the point differentials for the four quarters are independent. Assume that this model is correct.

(a) What is the probability that the home team wins?

(b) What is the conditional probability that the home team wins, given that it is behind by 5 points at halftime?

(c) What is the conditional probability that the home team wins, given that it is ahead by 5 points at the end of the first quarter?

Q10) Let X and Y be independent uniform $(0, 1)$ random variables.

(a) Find the joint density of $U = X$,$V = X + Y$.

(b) Compute the density function of V.

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Name	**PMF or PDF**	**Mean**	**Variance**
Ber(p)	$\mathbb{P}(X=1)=p, \mathbb{P}(X=0)=1-p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$ for $x=0,1,\ldots,n$	np	$np(1-p)$
Geom(p)	$p(1-p)^{x-1}$ for $x=1,2,\ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\binom{x-1}{r-1}p^r q^{x-r}$ for $x=r,r+1,\ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
HyperGeom(N,S,n)	$\frac{\binom{S}{x}\binom{N-S}{n-x}}{\binom{N}{n}}$ for $\max\{0,n-N+S\}\le x\le \min\{n,S\}$	$\frac{nm}{N}$	$n\cdot\frac{S}{N}\cdot\left(1-\frac{S}{N}\right)\cdot\frac{N-n}{N-1}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}$ for $x=0,1,\ldots$	λ	λ
Uniform(a,b)	$\frac{1}{b-a}$ for $x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$ for $x\in(-\infty,\infty)$	μ	σ^2
Exp(λ)	$\lambda e^{-\lambda x}$ for $x>0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma(a,λ)	$\frac{\lambda^a x^{a-1}e^{-\lambda x}}{\Gamma(a)}$ for $x>0$	$\frac{a}{\lambda}$	$\frac{a}{\lambda^2}$

TABLE 1. Table of Important Distributions to be provided in the Exams.

Geometric series	$\sum_{k=0}^{N} ap^k = \frac{a(1-p^{N+1})}{1-p}$ for $p\in(0,1)$
Power series for e^x	$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$
Gamma function	$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx$

TABLE 2. Table of formulas to be provided in the Exams.