$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

Supplementary Problems - Ch. 7 and Google PageRank

The following problems are sample problems meant to show the type of problems that might appear on the final exam for the sections after Exam 3, namely Google PageRank and the content of Chapter 7 (symmetric matrices, spectral theorem, quadratic forms, constrained optimization, and Singular Value Decomposition).

WARNING: The problems are **not** meant to be an exhaustive list. Please be aware that these are not necessarily the same problems that will appear on the final exam, and these problems may not cover every aspect of what may be asked. These are meant only to be a sample of problems that could appear.

1. (a) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

& positive semidefinite
Not invertible $\Rightarrow$ $\exists\, \lambda = 0$. $\Rightarrow$ smallest $= 0$

true	false	
○	⊗	If A is a singular symmetric matrix then the minimum value (= smallest eigenvalue) of the quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ subject to the constraint $\|\vec{x}\| = 1$ is zero.
○	⊗	Suppose the least of the singular values of A is 2 ($\min \sigma_i = 2$). Then, the quadratic form $Q(\vec{x}) = \|A\vec{x}\|^2 - \|\vec{x}\|^2$ is ~~indefinite~~ positive definite.
○	⊗	If $Q : \mathbb{R}^3 \to \mathbb{R}$ is a quadratic form having *only* cross terms, then it is indefinite.
○	⊗	If the singular values of a square $n \times n$ matrix A are $\sigma_1, \sigma_2, \ldots, \sigma_n$, then $\det(A) = \sigma_1\sigma_2\cdots\sigma_n$.

$Q(x) = (Ax)^T \cdot Ax - x^T \cdot x$
$= x^T \cdot A^T \cdot A \cdot x - x^T \cdot I \cdot x$
$= x^T \cdot (A^TA - I)\, x$

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}^2$

$\det(A^T \cdot A) = \sigma_1^2 \cdots \sigma_n^2$
$\Rightarrow \det(A) = \pm\, \sigma_1 \cdots \sigma_n$

(b) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	⊗	A symmetric matrix A which is not diagonalizable.
○	⊗	An $m \times n$ matrix A that does not have a singular value decomposition.
○	⊗	An indefinite quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ such that $Q(\vec{x}) \geq 0$ for all inputs $\vec{x}$.
○	⊗	An invertible $n \times n$ matrix A such that the minimum value of $Q(\vec{x}) = \vec{x}^T A^T A\vec{x}$ (positive semidefinite) subject to the constraint $\|\vec{x}\| = 1$ is less than 0.

Supplementary Problems - Ch. 7 and Google PageRank. Your initials: ________________

You do not need to justify your reasoning for questions on this page.

2. Give an example of a symmetric 3×3 matrix A such that:

   - the corresponding quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ is negative semi-definite and has no cross product terms;
   - the range of the corresponding linear transformation $T(\vec{x}) = A\vec{x}$ is 2-dimensional; and
   - one of the eigenvalues of A has algebraic multiplicity equal to 2.

$$A = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}$$

3. Find the minimum value of $Q(\vec{x}) = 4x_2x_3$ on the unit sphere in $\mathbb{R}^3$.

$$\boxed{-2}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = PDP^{-1}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \underline{\underline{-2}} \end{bmatrix}$$

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You do not need to justify your reasoning for questions on this page.

4. Let $A = U\Sigma V^T$, where

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}.$$

$\|Ax\| = \|U\Sigma V^T x\|$

$= \|\Sigma \cdot (V^T x)\|$

under $\|x\| = 1 = \|V^T x\|$, $= y$

$\Downarrow$

$\max \|\Sigma y\| = \sqrt{6}$ under $\|y\| = 1$.

(a) What is the rank of A? $\boxed{2}$

(b) $\max_{\|\vec{x}\|=1} \|A\vec{x}\| = \boxed{\sqrt{6}}$ ~~4~~

(c) $\det(A^T A) = \boxed{0}$

(d) What are the eigenvalues of $A^T A$? $\boxed{6, 4, 0}$

$\sqrt{\|Ax\|^2} = \sqrt{x^T \cdot (A^T A) \cdot x} = \sqrt{(\sqrt{6})^2} = \sqrt{6}$. $\Sigma y = \begin{bmatrix} \sqrt{6} \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(e) Find an orthonormal basis for $\mathrm{Row}(A)$.

$= \mathrm{Col}(A^T)$

$= \mathrm{Nul}(A)^\perp$

(f) Find an orthonormal basis for $\mathrm{Nul}(A^T)$.

$= \mathrm{Col}(A)^\perp$

5. ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***
   Consider the quadratic form $Q(x_1, x_2, x_3) = 3x_1^2 + x_2^2 + x_3^2 - 6x_2x_3$.

   (a) There is a change of variables $\vec{x} = P\vec{y}$, where P is an orthogonal matrix, such that the resulting quadratic form $Q(y_1, y_2, y_3)$ has no cross product terms. What is $Q(y_1, y_2, y_3)$ where $\vec{x} = P\vec{y}$?

   $$Q(y_1, y_2, y_3) = \boxed{3y_1^2 + 4y_2^2 - 2y_3^2}$$

   $$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

   (b) Is Q positive-definite, negative-definite, or indefinite? *Explain in a few words.*

   ○ positive-definite ○ negative-definite ☒ indefinite

Extra problem: in case you want to practice doing least-squares with non-linear models.

6. ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

In this problem, you will use the least-squares method to find the values α and β which best fit the curve

$$y = \alpha x^3 + \beta x^2$$

to the data points $(-1, 2), (0, 1), (1, 4)$ using the parameters α and β.

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = b$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\boxed{\left[\begin{array}{cc|c} -1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 1 & 4 \end{array}\right]}$$

(ii) What is the augmented matrix for the normal equations for this system.

$$A^T A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = A^T b$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

(iii) Find a least-squares solution to the linear system from (i) to determine the parameters α and β of the best fitting curve.

$\alpha = \boxed{1}$ $\beta = \boxed{3}$