Homework 3 Solution

Math 461: Probability Theory, Spring 2022
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Due date: Feb 11, 2022

1. A die is rolled until either 3 or 5 appears. Find the probability that a 5 occurs first. Simplify the answer.
   Hint: Let E_n denote the event that a 5 occurs on the n-th roll and no 3 or 5 occurs on the first $n-1$ rolls. Find $\mathbb{P}(E_n)$ and express the above probability in terms of them.

Solution: Following the hint we let E_n be the event that a 5 occurs on the n-th roll and no three or five occurs on the $(n-1)$ rolls up to that point. Then

$$\mathbb{P}(E_n) = \frac{4^{n-1} \cdot 1}{6^n},$$

as for n rolls total number of possible outcomes are 6^n and out of them E_n has 4^{n-1} many outcomes. Since we want the probability that a five comes first, this can happen at roll number one ($n = 1$), at roll number two ($n = 2$) or any subsequent roll. Moreover, the events E_n are mutually exclusive. Thus the probability that a five comes first is given by

$$\mathbb{P}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathbb{P}(E_n) = \sum_{n=1}^{\infty} \frac{4^{n-1}}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{4 \cdot 3^n} = \frac{1}{4} \cdot \frac{2/3}{1-2/3} = \frac{1}{2} = 0.5.$$

2. An urn contains 4 red and 8 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

Solution:

$$\begin{aligned}
\mathbb{P}(A \text{ wins in one move}) &= \frac{4}{12} = \frac{1}{3} \\
\mathbb{P}(A \text{ wins in three moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{4}{10} = \frac{56}{330} \\
\mathbb{P}(A \text{ wins in five moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} = \frac{7}{99} \\
\mathbb{P}(A \text{ wins in seven moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{4}{6} = \frac{2}{99} \\
\mathbb{P}(A \text{ wins in nine moves}) &= \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9} \cdot \frac{4}{8} \cdot \frac{3}{7} \cdot \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4} = \frac{2}{990} \\
\mathbb{P}(A \text{ wins}) &= \frac{1}{3} + \frac{56}{330} + \frac{7}{99} + \frac{2}{99} + \frac{2}{990} = \frac{59}{99} = 0.596.
\end{aligned}$$

3. Two fair dice are rolled. What is the conditional probability that none lands on 6 given that the dice land on different numbers?

Solution: Let E be the event that no dice lands on six, and let F be the event that the dice land of different numbers. Then $\mathbb{P}(EF) = \frac{5}{6} \cdot \frac{4}{6} = \frac{5}{9}$ and $\mathbb{P}(F) = \frac{6}{6} \cdot \frac{5}{6} = \frac{5}{6}$. Hence,

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(EF)}{\mathbb{P}(F)} = \frac{5/9}{5/6} = \frac{2}{3}.$$

4. Consider an urn containing 15 balls, of which 8 are red, 5 are green and 2 are blue. A sample of size 4 is to be drawn with replacement (without replacement). What is the conditional probability (**in each case**) that the first and third balls drawn will be red given that the sample drawn contains exactly 2 red balls?

Solution: Let A be the event that the sample drawn contains exactly 2 red balls. Let B be the event that the first and third ball drawn are red.
WITH REPLACEMENT: $\mathbb{P}(A) = \binom{4}{2}\frac{8^2 \cdot 7^2}{15^4}$ and $\mathbb{P}(AB) = \frac{8^2 \cdot 7^2}{15^4}$, hence $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \frac{1}{6}$.
WITHOUT REPLACEMENT: $\mathbb{P}(A) = \binom{4}{2}\frac{8 \cdot 7 \cdot 7 \cdot 6}{15 \cdot 14 \cdot 13 \cdot 12}$ and $\mathbb{P}(AB) = \frac{8 \cdot 7 \cdot 7 \cdot 6}{15 \cdot 14 \cdot 13 \cdot 12}$, hence $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(AB)}{\mathbb{P}(A)} = \frac{1}{6}$.

5. A closet contains 12 pairs of shoes. If 7 shoes are randomly selected without replacement, find the probability that there will be (a) at least one complete pair? (b) exactly 2 complete pairs? (c) exactly 2 complete pairs given that there is at least one complete pair.

Solution: Let A be the event that there are no complete pairs and B be the event that there is exactly one complete pair. Then
(a) $\mathbb{P}(A) = 1 - \frac{2^7\binom{12}{7}}{\binom{24}{7}}$.
(b) $\mathbb{P}(B) = \frac{2^3 \cdot \binom{12}{2} \cdot \binom{10}{3}}{\binom{24}{7}}$.
(c) $\mathbb{P}(B \mid A) = \frac{\mathbb{P}(BA)}{\mathbb{P}(A)} = \frac{\mathbb{P}(B)}{1 - \mathbb{P}(A)} = \frac{2^3 \cdot \binom{12}{2} \cdot \binom{10}{3}}{\binom{24}{7} - 2^7\binom{12}{7}}$.

6. Fifty-two percent of the students at a certain college are females. Five percent of the students in this college are majoring in computer science. Two percent of the students are women majoring in computer science. If a student is selected at random, find the conditional probability that
(a) the student is female given that the student is majoring in computer science;
(b) this student is majoring in computer science given that the student is female.

Solution:

(a) $\mathbb{P}(F \mid C) = \frac{\mathbb{P}(FC)}{\mathbb{P}(C)} = \frac{.02}{.05} = \frac{2}{5}$.

(b) $\mathbb{P}(C \mid F) = \frac{\mathbb{P}(FC)}{\mathbb{P}(F)} = \frac{.02}{.52} = \frac{1}{26}$.

7. Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Solution: Let E_i be the event that the ball drawn from the i-th urn is white, for $i = 1, 2, 3$ (for urn

A, B, C). Let F be the event that exactly two white balls were drawn. Then

$$\mathbb{P}(E_1 \mid F) = \frac{\mathbb{P}(E_1 F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E_1 E_2 E_3^c) + \mathbb{P}(E_1 E_2^c E_3)}{\mathbb{P}(E_1 E_2 E_3^c) + \mathbb{P}(E_1 E_2^c E_3) + \mathbb{P}(E_1^c E_2 E_3)} = \frac{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4}}{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4} + \frac{4\cdot 8\cdot 1}{6\cdot 12\cdot 4}} = \frac{7}{11}.$$

8. Urn I contains 2 white and 4 red balls, whereas urn II contains 1 white and 1 red ball. A ball is randomly chosen from urn I and put into urn II, and a ball is then randomly selected from urn II. What is
(a) the probability that the ball selected from urn II is white?
(b) the conditional probability that the transferred ball was white given that a white ball is selected from urn II?

Solution: (a) Let A be the event that the ball selected from urn I is white and let B be the event that the ball selected from urn II is white. The we have

$$\mathbb{P}(A) = 2/(2+4) = 1/3, \ \mathbb{P}(A^c) = 4/(2+4) = 2/3.$$

If the selected ball from urn I is white that after the transfer there are 2 white balls and 1 ed ball in urn II, thus $\mathbb{P}(B \mid A) = 2/(2+1) = 2/3$. Similarly we have $\mathbb{P}(B \mid A^c) = 1/3$. Thus we have by LoTP

$$\mathbb{P}(B) = \mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c) = (2/3)(1/3) + (1/3)(2/3) = 4/9.$$

(b) Using Bayes' rule we have

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B \mid A)\,\mathbb{P}(A) + \mathbb{P}(B \mid A^c)\,\mathbb{P}(A^c)} = \frac{(2/3)(1/3)}{(2/3)(1/3) + (1/3)(2/3)} = 1/2.$$

9. Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random.
(a) What is the probability that the marble is black?
(b) What is the probability that the first box was the one selected given that the marble is white?

Solution: (a) Let C be the event that the chosen box is box 1 containing 1 black and 1 white marble. Let B be the event that the chosen marble is black. (a) We want $\mathbb{P}(B)$. Using LoTP we have

$$\mathbb{P}(B) = \mathbb{P}(B \mid C)\,\mathbb{P}(C) + \mathbb{P}(B \mid C^c)\,\mathbb{P}(C^c) = \frac{1}{2}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} = \frac{7}{12} \approx 0.583.$$

(b) We want $\mathbb{P}(C \mid B^c)$. Using Bayes' rule we have

$$\mathbb{P}(C \mid B^c) = \frac{\mathbb{P}(B^c \mid C)\,\mathbb{P}(C)}{\mathbb{P}(B^c)} = \frac{1/2\cdot 1/2}{1-\mathbb{P}(B)} = \frac{1/4}{1-7/12} = \frac{3}{5} = 0.6.$$

10. Suppose that you continually collect coupons and that there are m different types. Suppose also that each time a new coupon is obtained, it is a type i coupon with probability $p_i, i = 1, 2, \ldots, m$. Suppose that you have just collected your n-th coupon. What is the probability that it is a new type?
Hint: Condition on the type of the n-th coupon.

Solution: $\mathbb{P}(\text{new type}) = \sum_{i=1}^{m} p_i \cdot \mathbb{P}(\text{new} \mid \text{the } n\text{-th coupon is of type } i) = \sum_{i=1}^{m} p_i (1-p_i)^{n-1}$.