MATH 461 MIDTERM 1 PRACTICE

1. Exam info

(1) **Time and Place**: March 2, in class (1–1:50pm).

(2) **Coverage**: Sections 1–1.6, 2.2–5, 3.2–4, 4.1–8 (up to material covered in Feb 23 class)

(3) No book or notes are allowed. Using calculator is okay.

(4) **Types of problems** may include:

  (a) Stating definitions

  (b) Write the statements and give proofs for theorems (or principles) covered in class

  (c) True and False

  (d) Giving examples

  (e) Computations for concrete examples

2. Review

2.1. Basic Combinatorics.

(1) Know how to compute the following number of ways and apply it to concrete examples:

  (a) Number of permutations (i.e. orderings) of n distinct objects

  (b) Number of permutations of n objects in which n_1 are alike, n_2 are alike, $\cdots$ and n_r are alike

  (c) Number of different ways to select k objects out of n distinct objects

  (d) Number of subsets of k elements of a set with n elements

  (e) Number of ways to split n distinct objects in r groups, with $n_1, n_2, \cdots, n_r$ objects in each group

  (f) Number of distinct, strictly positive integer solutions of $x_1 + x_2 + \cdots + x_r = n$

  (g) Number of distinct, non-negative integer solutions of $x_1 + x_2 + \cdots + x_r = n$

(2) Statement of Binomial Theorem

(3) Statement of Multinomial Theorem

(4) Know how to show some combinatoric identities:

  (a) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} =?$

  (b) $\binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \cdots + \binom{n-1}{k-1} = \binom{n}{k}$

  (c) $\binom{n_1}{0}\binom{n_2}{k} + \binom{n_1}{1}\binom{n_2}{k-1} + \cdots + \binom{n_1}{k}\binom{n_2}{0} = \binom{n_1+n_2}{k}$

2.2. Axiom of Probability.

(1) Definitions

   (a) Sample spaces
   (b) Events
   (c) Conditional probability
   (d) Independence (for 2 event, 3 events, and more than 3 events)
(2) Statement of Axiom of Probability
(3) Basic properties of probability
   (a) $\mathbb{P}(E^c) = 1 - \mathbb{P}(E)$
   (b) If $E \subset F$ then $\mathbb{P}(E) \leq \mathbb{P}(F)$.
   (c) $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
(4) Statement of Inclusion-Exclusion Principle
(5) Conditioning (multiplication rule): $\mathbb{P}(E \cap F) = \mathbb{P}(E|F)\mathbb{P}(F)$. For more than 2 events?
(6) Bayes formula: statement and proof
(7) Law of Total Probability: statement and proof

2.3. **Random variables.**

(1) Definitions
   (a) Random variables
   (b) PMF
   (c) CDF
   (d) Expectation
   (e) Variance
   (f) Bernoulli random variable
   (g) Binomial random variable
(2) Find the mean and variance of Binomial random variables.
(3) Compute the expectation of $g(X)$.
(4) Show that $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.
(5) $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ and $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$. Know how to prove and apply it to concrete examples.

3. Practice problems

Q1) Consider a "thick" coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads?

Q2) Suppose A, B and C are events with $\mathbb{P}(A) = 0.43, \mathbb{P}(B) = 0.40, \mathbb{P}(C) = 0.32, \mathbb{P}(A \cap B) = 0.29, \mathbb{P}(A \cap C) = 0.22, \mathbb{P}(B \cap C) = 0.20$ and $\mathbb{P}(A \cap B \cap C) = 0.15$. Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

Q3) If $\mathbb{P}(A) = 0.4, \mathbb{P}(B) = 0.5, \mathbb{P}(B \mid A) = 0.75$, find $\mathbb{P}(A \mid B)$ and $\mathbb{P}(A \mid B^c)$.

Q4) During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. Are events a randomly selected customer bought potato chips and a randomly selected customer bought beer independent?

Q5) Seventy percent of the light aircraft that disappear while in flight in Neverland are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft that has just disappeared has an emergency locator. What is the probability that it will not be discovered?

Q6) Suppose X is a random variable taking values in $S = \{0, 1, 2, 3, \ldots\}$ with PMF $p(0) = \mathbb{P}(X = 0) = c$, $p(k) = \mathbb{P}(X = k) = \frac{1}{2^k k!}, k = 1, 2, 3, \ldots$. Find the value of c that would make this a valid probability model. Find $\mathbb{E}[X], \mathbb{E}[2^X]$.

Q7) A certain basketball player knows that on average he will successfully make 78% of his free throw attempts. Assuming all throw attempts are independent. What are the expectation and the variance of the number of successful throws in 1020 attempts.

Q8) In each of the following cases, is it appropriate to use Binomial model? If yes, what are the values of its parameters n and p (if known)? If no, explain why Binomial model is not appropriate or what model to use if you can.

(a) A fair coin is tossed 3 times. X = number of Heads.

(b) A box contains 40 parts, 10 of which are defective. A person takes 7 parts out of the box with replacement. X = number of defective parts selected.

(c) Seven members of the same family are tested for a particular food allergy. X = number of family members who are allergic to this particular food.

(d) The bus numbered 1 arrives timely at Transit plaza 80% of the time. X = number of buses that arrives late before you see the first timely bus.

(e) Suppose that 5% of tax returns have arithmetic errors. 25 tax returns are selected at random. X = number of arithmetic errors in those 25 tax returns.

Q9) Suppose you roll two 5 faced dice, with faces labeled 1,2,3,4,5, and each equally likely to appear on top. Let X denote the smaller of the two numbers that appear. If both dice show the same number, then X is equal to that common number. Find the PMF of X. Compute $\mathbb{P}(X \leq 3 | X > 1)$ and $\mathbb{E}[5X - 2]$.

Q10) Consider an urn containing 21 balls, of which 7 are red, 8 are green and 6 are blue. A sample of size 15 is to be drawn with (or without) replacement. What is the conditional probability that the first 5 balls drawn will be green given that the sample drawn contains exactly 6 green balls?

Name	PMF	Mean	Variance
Ber(p)	$\mathbb{P}(X=1)=p, \mathbb{P}(X=0)=1-p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{x}p^x(1-p)^{n-x}$ for $x=0,1,\dots,n$	np	$np(1-p)$
Geom(p)	$p(1-p)^{x-1}$ for $x=1,2,\dots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
NegBin(r,p)	$\binom{x-1}{r-1}p^r q^{x-r}$ for $x=r,r+1,\dots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}$ for $x=0,1,\dots$	λ	λ

TABLE 1. Table of Important Distributions to be provided in the Exams.

Geometric series	$\sum_{k=0}^{N} ap^k = \frac{a(1-p^{N+1})}{1-p}$ for $p \in (0,1)$
Power series for e^x	$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

TABLE 2. Table of formulas to be provided in the Exams.

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