Section 2.1 : Matrix Operations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Identity and zero matrices
2. Matrix algebra (sums and products, scalar multiplies, matrix powers)
3. Transpose of a matrix

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. **Apply** matrix algebra, the matrix transpose, and the zero and identity matrices, to **solve** and **analyze** matrix equations.

Definitions: Zero and Identity Matrices

1. A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad 0_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2. The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

Sums and Scalar Multiples

Suppose $A \in \mathbb{R}^{m\times n}$, and $a_{i,j}$ is the element of A in row i and column j.

1. If A and B are $m \times n$ matrices, then the elements of $A+B$ are $a_{i,j} + b_{i,j}$.
2. If $c \in \mathbb{R}$, then the elements of cA are $ca_{i,j}$.

Componentwisely.

For example, if

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + c \begin{bmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{bmatrix}$$

What are the values of c and k?

$1 + c \cdot 7 = 15$, $c = 2$

$6 + c \cdot k = 16$, $k = 5$

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

1. $A + 0_{m\times n} = A$
2. $(A + B) + C = A + (B + C)$ ← Accociative
3. $r(A + B) = rA + rB$ dist.
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

Due to Component-wise

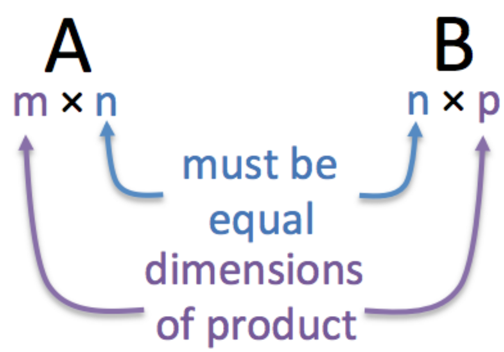
Matrix Multiplication

Definition

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{bmatrix} \in \mathbb{R}^{m \times p}$$

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

Row Column Method

If $A \in \mathbb{R}^{m\times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n\times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

Example

Compute the following using the row-column method.

$$C = AB = \underbrace{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}}_{2\times 2}^{\in \mathbb{R}^{2\times 2}} \begin{pmatrix} 3 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix}^{\in \mathbb{R}^{2\times 3}} \in \mathbb{R}^{2\times 3}$$

$$2 \times 2 = 2 \times 3$$

$$= \left[\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \end{bmatrix} \right]$$

$$2\cdot 3 + 0 \cdot 4$$

$$= \begin{bmatrix} [2 \;\; 0]\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & [2 \;\; 0]\cdot\begin{bmatrix} 0 \\ 5 \end{bmatrix} & [2 \;\; 0]\cdot\begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ [1 \;\; -1]\cdot\begin{bmatrix} 3 \\ 4 \end{bmatrix} & [1 \;\; -1]\cdot\begin{bmatrix} 0 \\ 5 \end{bmatrix} & [1 \;\; -1]\cdot\begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix}$$

Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B+C) = AB + AC$
3. (Right Distributive) $\cdots$ $(A+B)\cdot C = AC + BC$
4. (Identity for matrix multiplication) $I_m A = A I_n$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

$A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$

① $A\cdot B \in \mathbb{R}^{m\times p}$, $\underline{B\cdot A}$ well-defined?

② If $m = p$ $B\cdot A \in \mathbb{R}^{n\times n}$

$\shortparallel$

$A\cdot B \in \mathbb{R}^{m\times m}$

③ $m = n = p$ AB, $BA \in \mathbb{R}^{m\times m}$

in general $AB \neq BA$.

Ex

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq B \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$a, b \in \boxed{\mathbb{R}}$, $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

$A \cdot B = 0 \in \mathbb{R}^{m \times p} \Rightarrow$ $A \neq 0$, $B \neq 0$ is possible.

$$A \cdot \vec{x} = \vec{0}$$

$m \times n \cdot n \times 1$

$a, b, c \in \mathbb{R}$

$ab = a \cdot c \Rightarrow$ $a = 0$ or $b - c = 0$

$a(b-c) = 0$

$AB = A \cdot C \Rightarrow A \neq 0$ & $B \neq C$ is possible.

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$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \in \mathbb{R}^{m\times p}$$

In terms of linear transformations

$$\begin{tikzcd}
\mathbb{R}^p \arrow[r, bend left, "T(\vec{x}) = B\vec{x}"] \arrow[rr, bend right] & \mathbb{R}^n \arrow[r, bend left, "S(\vec{x}) = A\vec{x}"] & \mathbb{R}^m
\end{tikzcd}$$

$$S(T(\vec{x})) = S(B\vec{x}) = AB\vec{x}$$

Composition of Linear transformations
" = " Product of Matrices

- $2x = 3 \Rightarrow \frac{1}{2}\cdot(2x) = \frac{1}{2}3$

$$x = \frac{3}{2}$$

$$Ax = B$$

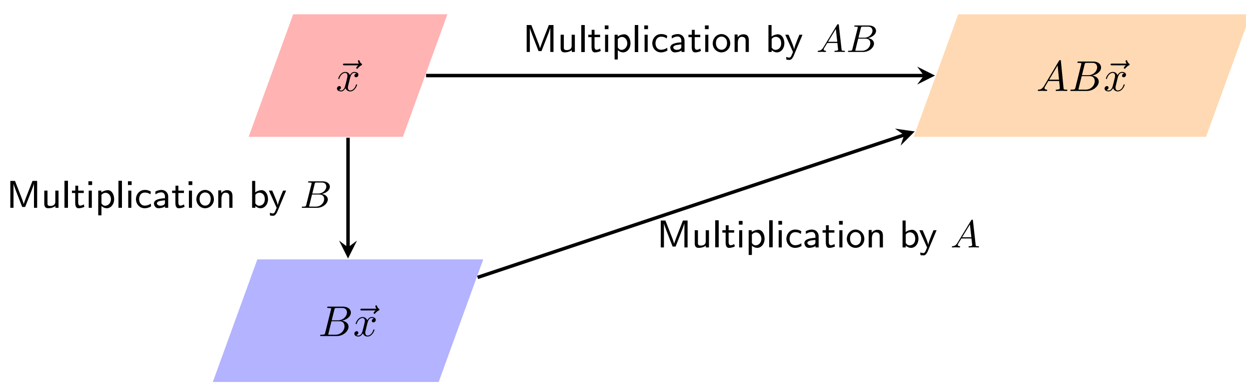
- $24 = 2^3 \times 3$ // $\underbrace{A} = \bigcirc \cdot \bigcirc \cdot \bigcirc$

The Associative Property

The associative property is $(AB)C = A(BC)$. If $C = \vec{x}$, then

$$(AB)\vec{x} = A(B\vec{x})$$

Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Example

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Give an example of a 2×2 matrix B that is non-commutative with A.

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n} \longrightarrow A^T \in \mathbb{R}^{n \times m}$

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 0 \\ 4 & 2 \\ 5 & 0 \end{bmatrix}$$

Properties of the Matrix Transpose

1. $(A^T)^T =$ A
2. $(A+B)^T =$ $A^T + B^T$
3. $(rA)^T =$ $r \cdot A^T$
4. $(AB)^T =$ $B^T \cdot A^T$

$A \in \mathbb{R}^{m \times n}$ $\quad B \in \mathbb{R}^{n \times p}$

$A \cdot B \in \mathbb{R}^{m \times p}$ $\quad (AB)^T \in \mathbb{R}^{p \times m}$

$\underbrace{\cancel{A^T}}_{n \times m} \cdot \underbrace{B^T}_{p \times n} \cdot \underbrace{A^T}_{n \times m}$

Matrix Powers

For any $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$\underbrace{A^k} = \underbrace{AA \dots A}_{k \text{ times}}$$

Example: Compute C^8.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 2^2 \end{bmatrix}$$

$$C^3 = C \cdot C^2 = C^2 \cdot C = \begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 2^3 \end{bmatrix}$$

$$\vdots$$

$$C^8 = \begin{bmatrix} 1^8 & 0 & 0 \\ 0 & 2^8 & 0 \\ 0 & 0 & 2^8 \end{bmatrix}$$

Example

Define

$$A = \underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \quad B = \underset{2\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}}, \quad C = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

Which of these operations are defined, and what are the dimensions of the result?

1. $\cancel{\underbrace{A}_{2\times 2} + 3\underbrace{C}_{3\times 3}}$
2. $\cancel{\underbrace{A}_{2\times 2}\underbrace{(AB)^T}_{3\times 2}}$
3. $\underbrace{A}_{2\times 2} + \underbrace{ABCB^T}_{2\times 2} \in \mathbb{R}^{2\times 2}$

$$\Big(\underset{2\times 2 \quad 2\times 3}{\overset{2\times 3}{AB}}\Big)^T$$

3×2

$$\underset{2\times 2}{A}\ \underset{2\times 3}{B}\ \underset{3\times 3}{C}\ \underset{3\times 2}{B^T} \in \mathbb{R}^{2\times 2}$$

Additional Examples

True or false:

1. For any I_n and any $A \in \mathbb{R}^{n\times n}$, $(I_n + A)(I_n - A) = I_n - A^2$.

True

Identity

$n \times n$

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$$= I_n \cdot I_n - \cancel{I_n \cdot A} + \cancel{A \cdot I_n} - A^2$$

$$= I_n - A^2$$

2. For any A and B in $\mathbb{R}^{n\times n}$, $(A+B)^2 = A^2 + B^2 + 2AB$.

False

$AB \neq BA$.

$$(A+B)(A+B)$$

$$A^2 + AB + BA + B^2$$

Ex

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 & 0 \\ 0 & 2\cdot 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$$

Section 2.2 : Inverse of a Matrix

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."

- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of the lecture we'll discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations.
2. Elementary matrices and their role in calculating the matrix inverse.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems.
2. Compute the inverse of an $n \times n$ matrix, and use it to solve linear systems.
3. Construct elementary matrices.

Motivating Question

Is there a matrix, A, such that $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} A = I_3$?

In this section $A \in \mathbb{R}^{n \times n}$

"Square"

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

$$AC = CA = I_n.$$

If there is, we write $C = A^{-1}$. "A inverse"

because C is unique.

The Inverse of a 2×2 Matrix

There's a formula for computing the inverse of a 2×2 matrix.

Theorem

invertible

The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular if and only if $\boxed{ad - bc} \neq 0$, and then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

State the inverse of the matrix below.

$$\begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$$

$$ad - bc = 2 \cdot (-7) - 5 \cdot (-3)$$
$$= 1 \neq 0$$

$$= \frac{1}{1} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$$

Solve

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

Exercise

The Matrix Inverse

non-singular

⇕

Invertible

⇕

Theorem

$A \in \mathbb{R}^{n\times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a ~~unique~~ solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

Example

Solve the linear system.

$$\begin{cases} 3x_1 + 4x_2 = 7 \\ 5x_1 + 6x_2 = 7 \end{cases}$$

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 7 \\ 7 \end{bmatrix}}_{\vec{b}}$$

$ad - bc$

$= 3 \cdot 6 - 4 \cdot 5$

$= -2 \neq 0$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \cdot \vec{b}$$

$$= \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} 14 \\ 14 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
3. $(A^T)^{-1} = (A^{-1})^T$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

$$\mathbb{R}^n \xrightarrow{B\vec{x}} \mathbb{R}^n \xrightarrow{A\vec{x}} \mathbb{R}^n$$

A has an inverse

$\Longleftrightarrow$ For all $\vec{b} \in \mathbb{R}^n$

$A\vec{x} = \vec{b}$ is consistent.

How to Find A^{-1} ?

$$\underline{A \cdot A^{-1}} = I_n$$

$$[A v_1 \quad A v_2 \quad \cdots \quad A v_n] = [e_1 \quad e_2 \quad \cdots \quad e_n]$$

$$A^{-1} = [v_1 \quad v_2 \quad \cdots \quad v_n]$$

$$\begin{cases} A v_1 = e_1 & [A \mid e_1] \to [I \mid v_1] \\ A v_2 = e_2 & [A \mid e_2] \to [I \mid v_2] \\ \quad \vdots & \quad \vdots \qquad\qquad \vdots \\ A v_n = e_n & [A \mid e_n] \end{cases}$$

$$[A \mid e_1 \quad e_2 \cdots e_n] = [A \mid I]$$

$$\downarrow$$

$$[I \mid \underbrace{v_1 \cdots v_n}_{= A^{-1}}]$$

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$A \in \mathbb{R}^{n \times n}$ $\qquad T: \mathbb{R}^n \to \mathbb{R}^n$

A invertible (non-singular)

def
$\Leftrightarrow$ There exists $C \in \mathbb{R}^{n \times n}$ such that $A \cdot C = I_n$
$(CA = I_n)$

$\Leftrightarrow$ For all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ $\left\{ \begin{array}{l} \text{has a (unique) solution} \\ \text{is consistent} \end{array} \right.$

$\Leftrightarrow$ T is onto $\Leftrightarrow$ ....

$\Leftrightarrow$ T is 1-1 $\Leftrightarrow$ ----.

$\Leftrightarrow$ $A \to I_n$ after row operations

How to find A^{-1} = Solve n linear systems

$$\left\{ \begin{array}{l} A v_1 = e_1 \\ \vdots \\ A v_n = e_n \end{array} \right.$$

$$[A \mid I_n] \longrightarrow [I_n \mid A^{-1}]$$

An Algorithm for Computing A^{-1}

If $A \in \mathbb{R}^{n\times n}$, and $n > 2$, how do we calculate A^{-1}? Here's an algorithm we can use:

1. Row reduce the augmented matrix $(A \,|\, I_n)$
2. If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 2R_3}]{} \left[\begin{array}{c|ccc} & 0 & 1 & -3 \\ I_n & 1 & 0 & -2 \\ & 0 & 0 & 1 \end{array}\right]$$

A invertible

$= A^{-1}$

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$\begin{aligned} A\vec{x}_1 &= \vec{e}_1 \\ A\vec{x}_2 &= \vec{e}_2 \\ &\vdots \\ A\vec{x}_n &= \vec{e}_n \end{aligned}$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Over the next few slides we explore another explanation for how our algorithm works. This other explanation uses elementary matrices.

A is invertible $\iff$ A is Row Equivalent to I_n

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

1. swap rows
2. multiply a row by a non-zero scalar
3. add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{swap } R_1, R_2]{} \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R_1 \to R_1 + 2R_3]{} \begin{bmatrix} 1+2\cdot7 & 2+2\cdot8 & 3+2\cdot9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Example

$R_2 \to R_2 + 2R_1$

Suppose

$$E \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By inspection, what is E? How does it compare to I_3?

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrix is a matrix from I_n by performing 1 row operation.

A invertible

$\Longleftrightarrow \quad A \longrightarrow \cdots \longrightarrow I_n$

$\Longleftrightarrow \quad E_k \cdots E_2 E_1 A = I_n$

$\Longleftrightarrow \quad A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ (product of elementary matrix)

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(\overbrace{E_k \cdots E_3 E_2 E_1}^{A^{-1}})A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

Theorem

Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x} = \vec{b}$$

We would calculate A^{-1} and then:

$$\vec{x} = A^{-1} \cdot \vec{b}$$

- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, doesn't that we **should**.

Section 2.3 : Invertible Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

Topics and Objectives

Topics
We will cover these topics in this section.

1. The invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced.

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize the invertibility of a matrix using the Invertible Matrix Theorem.
2. Construct and give examples of matrices that are/are not invertible.

Motivating Question
When is a square matrix invertible? Let me count the ways!

The Invertible Matrix Theorem (IMT)

Invertible matrices enjoy a rich set of equivalent descriptions.

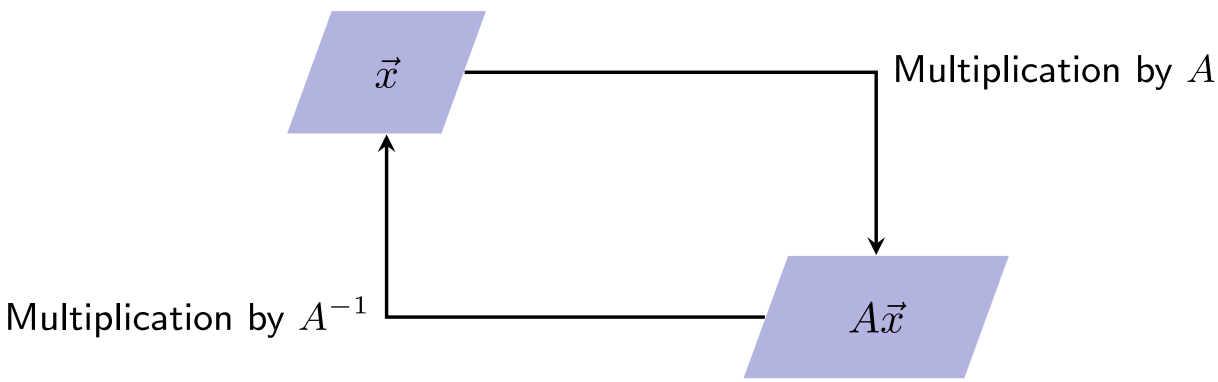
Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

a) *A is invertible.*
b) *A is row equivalent to I_n.*
c) *A has n pivotal columns. (All columns are pivotal.)*
d) *$A\vec{x} = \vec{0}$ has only the trivial solution.*
e) *The columns of A are linearly independent.*
f) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.*
g) *The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.*
h) *The columns of A span $\mathbb{R}^n$.*
i) *The linear transformation $\vec{x} \mapsto A\vec{x}$ is onto.*
j) *There is a $n \times n$ matrix C so that $CA = I_n$. (A has a left inverse.)*
k) *There is a $n \times n$ matrix D so that $AD = I_n$. (A has a right inverse.)*
l) *A^T is invertible.*

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.



The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} =$$

The Invertible Matrix Theorem: Final Notes

- Items j and k of the invertible matrix theorem (IMT) lead us directly to the following theorem.

> **Theorem**
>
> If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

Example 1

Is this matrix invertible?

$$\begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

invertible.

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix}$$

Not scalar multiple.

Example 2

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & * & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

A invertible

$\iff$ Rows are lin. indep.

Section 2.4 : Partitioned Matrices

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
- William Paul Thurston

Multiple perspectives of the same concept is a theme of this course; each perspective deepens our understanding. In this section we explore another way of representing matrices and their algebra that gives us another way of thinking about them.

Topics and Objectives

Topics

We will cover these topics in this section.

1. Partitioned matrices (or block matrices)

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Apply partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication.

What is a Partitioned Matrix?

Example

This matrix:

$$\left[\begin{array}{ccc|cc} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 4 & 2 \end{array}\right]$$

can also be written as:

$$\begin{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & 2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Another Example of a Partitioned Matrix

Example: The reduced echelon form of a matrix. We can use a partitioned matrix to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & 0 & * & \cdots & * \\ 0 & 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} I_4 & F \\ 0 & 0 \end{bmatrix}$$

This is useful when studying the **null space** of A, as we will see later in this course.

Solution "

Row Column Method

Recall that a row vector times a column vector (of the right dimensions) is a scalar. For example,

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1\cdot 1 + 1\cdot 0 + 1\cdot 2$$

This is the **row column** matrix multiplication method from Section 2.1.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\text{row}_i\, A \cdot \text{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Partitioned matrices can be multiplied using this method, as if each block were a scalar (provided each block has appropriate dimensions).

Example

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{R}^{2n \times 2n} \qquad A, B, C, D \in \mathbb{R}^{n\times n}$$

$PA + RB$ $\neq$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right] = \left[\begin{array}{c|c} AP + B\cdot R & AQ + BS \\ \hline CP + DR & CQ + DS \end{array}\right]$$

Example $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in \mathbb{R}^{2\times 2}$ $\quad \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}$

$ac = ac - b\cdot 0 \neq 0$

$\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \in \mathbb{R}^{2n\times 2n}$ $\quad \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \frac{1}{AC}\begin{bmatrix} C & -B \\ 0 & A \end{bmatrix}$

Example of Row Column Method

Recall, using our formula for a 2×2 matrix, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^{-1} = \frac{1}{ac}\begin{bmatrix} c & -b \\ 0 & a \end{bmatrix}$. $= \begin{bmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{bmatrix}$

Example: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{n\times n}$ are invertible matrices. Construct the inverse of $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & I_n \end{array}\right]$$

A invertible

$AP + B\overset{0}{R} = I_n \rightarrow AP = I_n \rightarrow \boxed{P = A^{-1}}$

$AQ + BS = 0$

$R = C^{-1}CR = C^{-1}0 \rightarrow \boxed{R = 0}$

$CS = I_n \rightarrow$ C invertible

$\boxed{S = C^{-1}}$

$A^{-1}A\cdot Q = A^{-1}(-BS) = A^{-1}(-BC^{-1})$

$\boxed{Q} = \boxed{-A^{-1}BC^{-1}}$ $\quad \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}$

Section 2.5 : Matrix Factorizations

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity." - Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop.

Topics and Objectives

Topics

We will cover these topics in this section.

1. The LU factorization of a matrix
2. Using the LU factorization to solve a system
3. Why the LU factorization works

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute an LU factorization of a matrix.
2. Apply the LU factorization to solve systems of equations.
3. Determine whether a matrix has an LU factorization.

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using

$$\vec{x} = A^{-1}\vec{b}$$

- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

$$A \longrightarrow \cdots \longrightarrow \text{REF}$$

$$\underbrace{E_k \cdots E_3 E_2 E_1}_{\text{Invertible}} A = \begin{bmatrix} & * \\ 0 & \end{bmatrix}$$

$$A = \underbrace{\left(E_1^{-1} \cdot E_2^{-1} \cdots E_k^{-1}\right)} \cdot \underbrace{\text{REF}}$$

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

$$A = L \cdot U$$

Triangular Matrices

lower position than diagonal

- A rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$.
  Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$.
  Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

Ask: Can you name a matrix that is both upper and lower triangular?

The LU Factorization

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an **echelon** form of A.

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

$$\underset{m\times n}{A} = \underset{m \times m}{\underbrace{L}} \cdot \underset{m\times n}{U}$$

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows. Then,

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. They happen to be lower triangular and invertible, e.g.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 2R_1$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

Using the LU Decomposition

Goal: given A and $\vec{b}$, solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm: construct $A = LU$, solve $A\vec{x} = LU\vec{x} = \vec{b}$ by:

1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
2. Backwards solve for x in $U\vec{x} = \vec{y}$.

Example: Solve the linear system whose LU decomposition is given.

$$\underset{\underset{\mathbb{R}^{4\times 3}}{\Uparrow}}{A} = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{=\vec{y}} = \vec{b}$$

$$\begin{cases} U\vec{x} = \vec{y} \longrightarrow \text{Solve for } \vec{x} \in \mathbb{R}^3 \\ L\vec{y} = \vec{b} \longrightarrow \text{Find } \vec{y} \in \mathbb{R}^4 \end{cases}$$

① $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$

$y_1 = 2$

$y_1 + y_2 = 3$, $y_2 = 1$

$2y_2 + y_3 = 2$, $y_3 = 0$

$y_3 + y_4 = 0$, $y_4 = 0$

Forward Solving

② $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$$x_1 = 2$$

$$x_3 + 2x_2 = 1 \qquad x_2 = \frac{1}{2}$$

$$x_3 = 0$$

Backward Solving.

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Note that

- In MATH 1554, the only row replacement operation we can use is to *replace a row with a multiple of a row above it*.
- More advanced linear algebra courses address this limitation.

Example: Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix}$$

$$\underbrace{E_3 E_2 E_1}\, A = U$$

$$A = \underbrace{(E_1^{-1} E_2^{-1} E_3^{-1})}_{= L} \cdot U$$

E_1: $R_2 \to R_2 + 4R_1$

E_2: $R_3 \to R_3 - 2R_1$

$$\longrightarrow \begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -10 & 12 \end{pmatrix}$$

E_3: $R_3 \to R_3 - 5R_2$

$$\longrightarrow \underbrace{\begin{pmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{pmatrix}} = U$$

$$L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}$$

Exercise

lower triangular

$$\underset{m \times n}{A} = \underset{m \times m}{L} \cdot \underset{m \times n}{U} \leftarrow \text{REF of } A \text{ only using replacement}$$

same size

Summary

$$E_k \cdots E_2 E_1 A = U$$

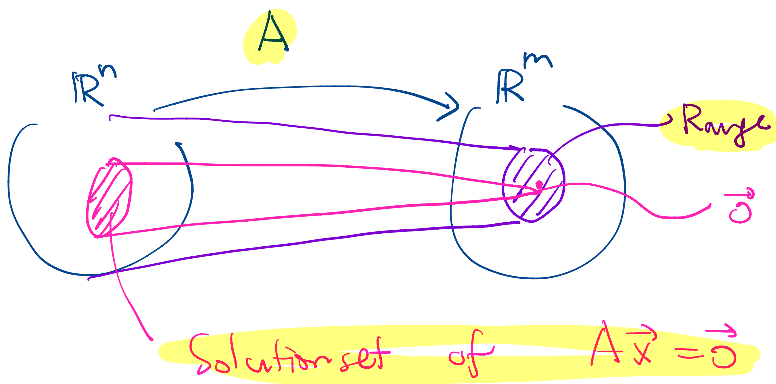
$$A = \underbrace{(E_k \cdots E_1)^{-1}}_{= L} \cdot U$$

- To solve $A\vec{x} = LU\vec{x} = \vec{b}$, ($U\vec{x} = \vec{y}$)
  1. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
  2. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.
- To compute the LU decomposition:
  1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
  2. Place entries in L such that the same sequence of row operations reduces L to I.
- The textbook offers a different explanation of how to construct the LU decomposition that students may find helpful.
- Another explanation on how to calculate the LU decomposition that students may find helpful is available from MIT OpenCourseWare: www.youtube.com/watch?v=rhNKncraJMk

Section 2.8 : Subspaces of $\mathbb{R}^n$

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics

We will cover these topics in this section.

1. Subspaces, Column space, and Null spaces
2. A basis for a subspace.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a set is a subspace.
2. Determine whether a vector is in a particular subspace, or find a vector in that subspace.
3. Construct a basis for a subspace (for example, a basis for Col(A))

Motivating Question

Given a matrix A, what is the set of vectors $\vec{b}$ for which we can solve $A\vec{x} = \vec{b}$?

Subsets of $\mathbb{R}^n$

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

No structure

Interested in a collection of vectors with Structure

vector addition

Scalar multiple.

= Subspace

Subspaces in $\mathbb{R}^n$

Definition

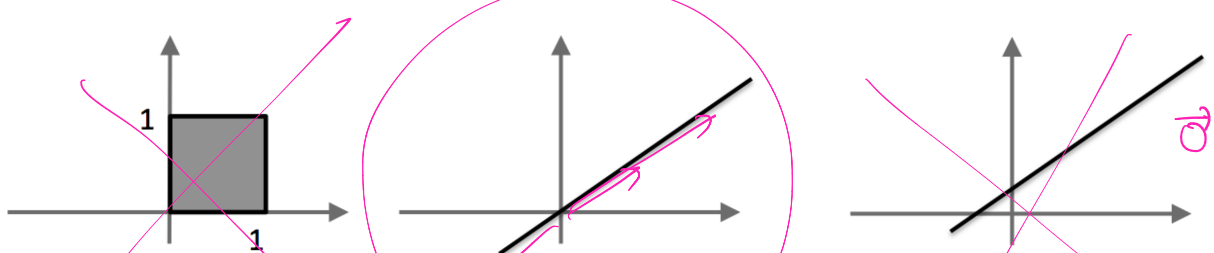
A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\vec{u} \in H$
2. $\vec{u} + \vec{v} \in H$

In particular, $\vec{0} \in H$

Note that condition 1 implies that the zero vector must be in H.

Example 1: Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square

b) a line passing through the origin

c) a line that doesn't pass through the origin

$\vec{0} \notin H$.

$(0,2) = 2\cdot(0,1)$

the smallest subspace containing $v_1, \ldots, v_p$.

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$, that $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is:

= Set of All Linear Combi.

This is a **subspace**, spanned by $\vec{v}_1, \ldots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

1. The **column space of** A, $\text{Col}\, A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \ldots, \vec{a}_n$.
2. The **null space of** A, $\text{Null}\, A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

- Col(A) = Range of T
- Why is Null (A) a subspace?

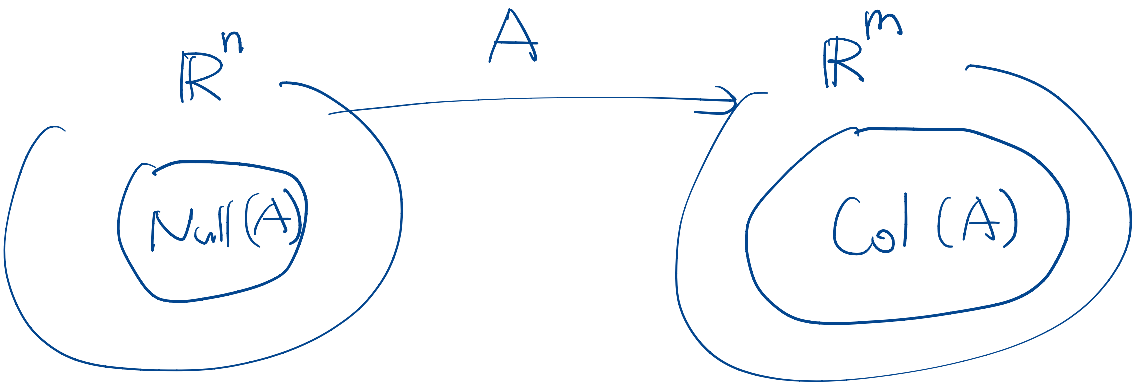
Pf: $\vec{x}, \vec{y} \in \text{Null}(A)$

$A\vec{x} = 0$

$A\vec{y} = 0$

$A(c\vec{x} + d\vec{y}) = c(A\vec{x}) + d(A\vec{y}) = \vec{0}$

$c\vec{x} + d\vec{y} \in \text{Null}(A)$ ⌟



Example

Is $\vec{b}$ in the column space of A?

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & \cancel{-4} & 3 \\ -4 & 6 & \cancel{-2} & 3 \\ -3 & 7 & \cancel{6} & -4 \end{array}\right] \longrightarrow \quad \text{Exercise.}$$

Note

- Row operation changes Column Space
- Col (A) is related to Pivot Columns
- Null (A) stays along row operations.

$$\text{Null}(A) = \text{Null}\begin{pmatrix} 1 & -3 & -4 \\ 0 & \cancel{-6}\ 1 & \cancel{-18}\ 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{Null}\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + 5x_3 = 0$

$x_2 + 3x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5x_3 \\ -3x_3 \\ x_3 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$$

$$\text{Null}(A) = \text{Span}\left(\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\} \right)$$

Example 2 (continued)

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$\vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

$H = \text{Span}(\{v_1, \dots, v_k\})$

Basis

lin. indep

Basis

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

The set $H = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \mid x_1 + 2x_2 + x_3 + 5x_4 = 0 \}$ is a subspace.

" Null ([1 2 1 5])

a) H is a null space for what matrix A?

b) Construct a basis for H.

$x_1 = -2x_2 - x_3 - 5x_4$

$$\begin{bmatrix} -2x_2 - x_3 - 5x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\text{Col}(A) = \text{Span}(\{\text{Cols in } A\})$

$\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$

$\uparrow$ $m \times n$

Basis of H: the set of vectors in H

(i) $\{v_1, \cdots, v_k\}$ lin. indep

(ii) $\text{Span}(\{v_1, \cdots, v_k\}) = H$

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{bmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 2x_2 = 0$

$x_3 = 0$

$\Rightarrow \begin{cases} x_1 = 2x_2 \\ x_3 = 0 \end{cases}$

$$\text{Basis for Col}(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

$$\text{Basis for Null}(A) = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Additional Example

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$.

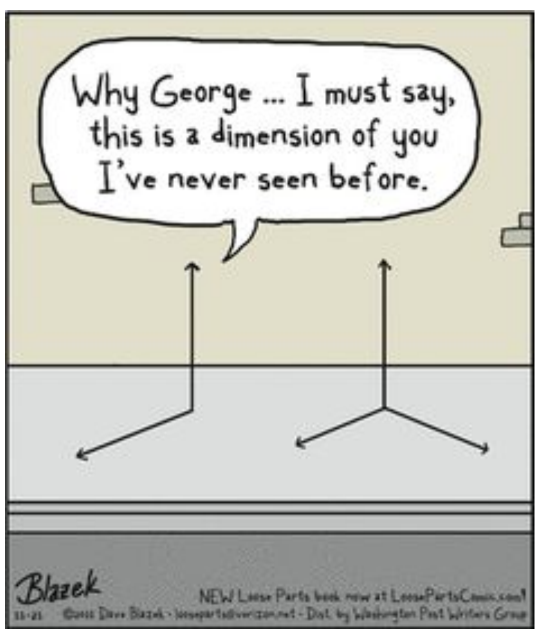
1. Give an example of a vector that is in V. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
2. Give an example of a vector that is not in V. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
3. Is the zero vector in V?
4. Is V a subspace?

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V$$

Section 2.9 : Dimension and Rank

Chapter 2 : Matrix Algebra

Math 1554 Linear Algebra



Topics and Objectives

Topics
We will cover these topics in this section.

1. Coordinates, relative to a basis.
2. Dimension of a subspace.
3. The Rank of a matrix

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Calculate the coordinates of a vector in a given basis.
2. Characterize a subspace using the concept of dimension (or cardinality).
3. Characterize a matrix using the concepts of rank, column space, null space.
4. Apply the Rank, Basis, and Matrix Invertibility theorems to describe matrices and subspaces.

<u>Example</u> $\mathbb{R}^2$ is a subspace

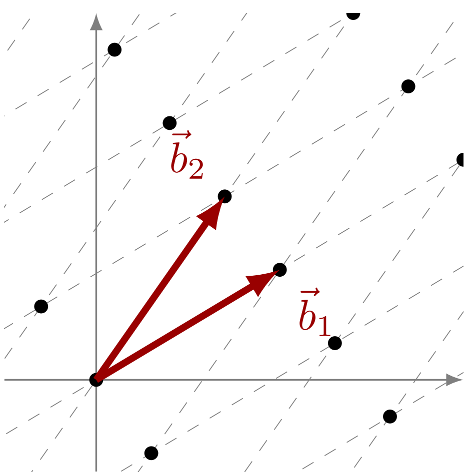
$\mathcal{B}_1 = \{ \vec{e}_1 , \vec{e}_2 \}$, where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

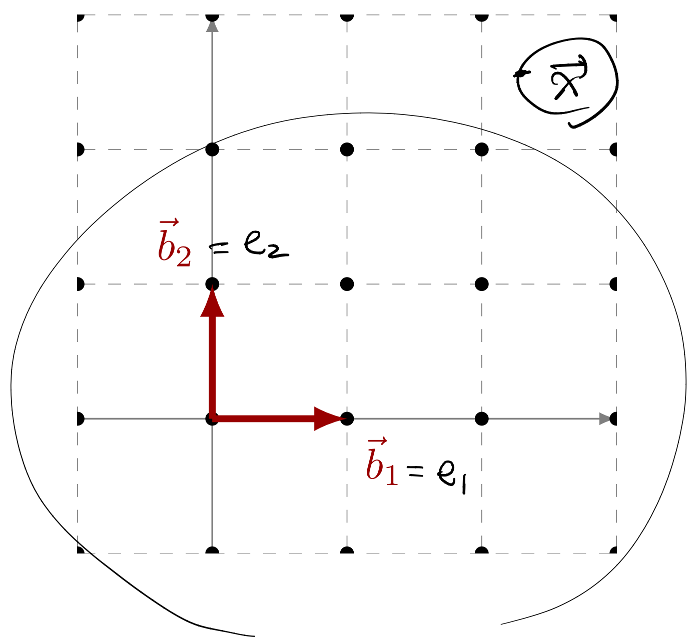
$\mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

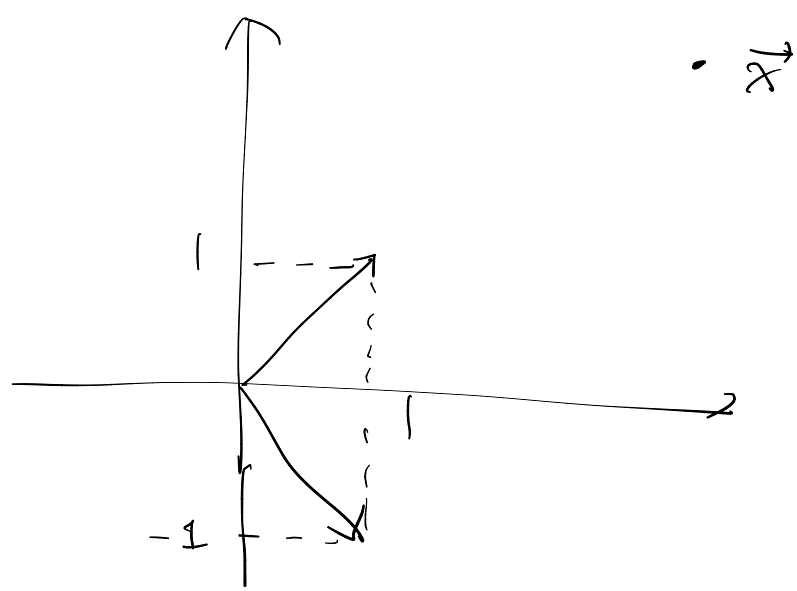
Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.







Coordinates

Definition

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \ldots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinate vector of $\vec{x}$**

$\mathcal{B}$ is a basis for H

(i) Span $(\mathcal{B}) = H$

(ii) $\mathcal{B}$ is lin. indep.

From (i) : $\forall x \in H$ can be written as

$$\vec{x} = c_1 \vec{b_1} + \cdots + c_p \vec{b_p}$$

From (ii) $\Rightarrow$ $-$ $\vec{x} = d_1 \vec{b_1} + \cdots + d_p \vec{b_p}$

$$\vec{0} = (c_1 - d_1)\vec{b_1} + \cdots + (c_p - d_p)\vec{b_p}$$

Example 1

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

$$\vec{x} = 2 \cdot v_1 + 3 v_2$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Dimension

zero subspace $= \{\vec{0}\}$

$\neq \phi$

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{0\} = 0$.

Theorem

Any two choices of bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of a non-zero subspace H have the same dimension.

Examples:

1. $\dim \mathbb{R}^n =$ n

   $[1 \ 1 \ \cdots \ 1] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$

2. $H = \{(x_1, \ldots, x_n) \ : \ x_1 + \cdots + x_n = 0\}$ has dimension $n-1$.

   $= \text{Null}([1 \ 1 \ \cdots \ 1])$

3. $\underline{\dim(\text{Null}\, A)}$ is the number of free variables.
4. $\dim(\text{Col}\, A)$ is the number of pivot columns.

$n = \dim(\text{Null}(A)) + \dim(\text{Col}(A)) = \#$ of Columns

$A \in \mathbb{R}^{m \times n}$

Q: P/I

$A \in \mathbb{R}^{5\times 3}$ $\dim(\text{Col}(A)) = 4$

Rank

$\text{rank}(A) = \dim(\text{Col}(A))$

Definition

The **rank** of a matrix A is the dimension of its column space.

Example 2: Compute rank(A) and dim(Nul(A)).

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 3$

$\dim(\text{Null}(A)) = 2$

Rank, Basis, and Invertibility Theorems

Theorem (Rank Theorem)

If a matrix A has n columns, then $\mathrm{Rank}\, A + \dim(\mathrm{Nul}\, A) = n$*.*

Theorem (Basis Theorem)

Any two bases for a subspace have the same dimension.

Theorem (Invertibility Theorem)

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. *A is invertible.*
2. *The columns of A are a basis for $\mathbb{R}^n$.*
3. $\mathrm{Col}\, A = \mathbb{R}^n$.
4. $\mathrm{rank}\, A = \dim(\mathrm{Col}\, A) = n$.
5. $\mathrm{Null}\, A = \{0\}$.

Examples

If possible give an example of a 2×3 matrix A, that is in RREF and has the given properties.

a) rank$(A) = 3$ NP.

b) rank$(A) = 2$ $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$

c) dim(Null$(A)) = 2$ $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$

d) Null$A = \{0\}$ NP

In general $A \in \mathbb{R}^{m \times n}$

$\underline{\text{Null}(A) = \{0\}}$ $\iff$ T is 1-1

$\iff$ $A\vec{x} = \vec{0}$ has unique trivial sol.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

3. (2 points) Let $\mathcal{H}$ be a subspace of $\mathbb{R}^3$ that is composed of all vectors $\vec{x} = (x_1, x_2, x_3)$ that satisfy the following two equations:

$$x_1 + 3x_2 - x_3 = 0$$
$$2x_1 + 5x_2 + x_3 = 0$$

What is the dimension of $\mathcal{H}$?

$$\dim\mathcal{H} = \boxed{1}$$

$\mathcal{H} = \text{Null}(A)$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 5 & 1 \end{bmatrix}$$

4. (2 points) Let $\mathcal{V}$ be a subspace of $\mathbb{R}^3$ that is spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right\}$$

What is the dimension of $\mathcal{V}$?

$$\dim\mathcal{V} = \boxed{3}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 0 & 3 & -3 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} & & & \end{bmatrix}$$

$\mathcal{V} = \text{Col}(A)$

$\dim(\mathcal{V}) = \#$ of Pivots