MATH 461 LECTURE NOTE
WEEK 7

DAESUNG KIM

1. Intro to Continuous Random Variables (Sec 5.1)

A discrete random variable is a random variable whose possible values are countable. However, one can think a random variable with uncountable possible values, for instance, the lifetime of a light bulb. For a discrete random variable, we assign the probabilities for each possible values, which is not possible for the case with uncountable possible values. Then, how can we define the probabilities for this case?

Definition
A random variable X is a continuous random variable if there is a nonnegative function f on $\mathbb{R}$ such that

$$\mathbb{P}(X \in B) = \int_B f(x)\,dx$$

for any set $B \subseteq \mathbb{R}$. The function f is called the probability density function of X.

In particular, we have

$$\mathbb{P}(X \in \mathbb{R}) = \int_{\mathbb{R}} f(x)\,dx = 1,$$
$$\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx,$$
$$\mathbb{P}(X = a) = \int_a^a f(x)\,dx = 0.$$

Example 1. Suppose that X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} C(3x - x^2), & 0 < x < 3, \\ 0, & \text{otherwise.} \end{cases}$$

Find C and $\mathbb{P}(X > 1)$.

The cumulative distribution function (in short, cdf) of a random variable X is defined by

$$F(a) = \mathbb{P}(X \le a) = \mathbb{P}(X < a) = \int_{-\infty}^a f(x)\,dx.$$

That is, the cdf is the integral of the probability density of X. Note that the cdf is continuous (regardless of the continuity of f). On the other hands, if we differentiate $F(a)$ with respect to a, then

$$\frac{d}{da}F(a) = \frac{d}{da}\int_{-\infty}^a f(x)\,dx = f(a).$$

That is, the probability density is the derivative of the cdf. Note that, for small $\varepsilon > 0$, we have

$$\mathbb{P}\left(a - \frac{\varepsilon}{2} \le X \le a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)\,dx \approx \varepsilon f(a).$$

Example 2. Let X be a continuous random variable with the density f_X and the cdf F_X. Let $Y = X^2$. What is the density of Y?

2. Expectation and Variance of Continuous RVs (Sec 5.2)

Expectation
Let X be a continuous random variable with the density f. The expectation (or mean, or the expected value) of X is defined by

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x)\, dx.$$

Example 3. Find $E[X]$ when the density function of X is given by

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 4. Let X be a random variable with density

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = e^X$.

(i) Find the density f_Y and distribution function F_Y of Y.
(ii) Compute $\mathbb{E}[Y]$.

Proposition 5. *Let X be a continuous random variable with the density f.*

(i) *If $X \ge 0$, then*

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, dx.$$

(ii) *If g a real-valued function, then $\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x) f(x)\, dx$. In particular $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$ for $a, b \in \mathbb{R}$.*

Proof. (i)

$$\begin{aligned} \int_0^\infty \mathbb{P}(X > x)\, dx &= \int_0^\infty \int_x^\infty f(y)\, dy dx \\ &= \int_0^\infty \int_0^y f(y)\, dx dy \\ &= \int_0^\infty y f(y)\, dy \\ &= \mathbb{E}[X]. \end{aligned}$$

(ii) Suppose g is nonnegative. Since $Y = g(X)$ is nonnegative, it follows from (i) that

$$\begin{aligned} \mathbb{E}[g(X)] &= \int_0^\infty \mathbb{P}(g(X) > y)\, dy \\ &= \int_0^\infty \int_{x: g(x) > y} f(x)\, dx dy \\ &= \int \int_0^{g(x)} f(x)\, dy dx \\ &= \int g(x) f(x)\, dx. \end{aligned}$$

For general g, one can use

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > y)\, dy - \int_0^\infty \mathbb{P}(Y < -y)\, dy$$

(see [SR, p.215, Problem 5.2]).

□

Example 6 (Revisit)**.** Compute $\mathbb{E}[e^X]$ where X is a random variable with density

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 7. Consider an interval $[0,1]$ and a point $p \in [0,1]$. Let X be a random variable with density

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

and consider two subintervals $[0, X]$ and $[X, 1]$. Let L be the length of one of these subintervals that contains p.

(i) Find a function g such that $L = g(X)$.
(ii) Compute $\mathbb{E}[L]$.

Variance
The variance is defined by

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{\mathbb{R}} (x - \mathbb{E}[X])^2 f(x)\, dx.$$

References

[SR] Sheldon Ross, *A First Course in Probability*, 9th Edition, Pearson

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu