Section 7.1 : Diagonalization of Symmetric Matrices

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Symmetric matrices
2. Orthogonal diagonalization

Learning Objectives

1. Construct an orthogonal diagonalization of a symmetric matrix, $A = PDP^T$.

Symmetric Matrices

" square "

Definition

Matrix A is **symmetric** if $A^T = A$.

Example. Which of the following matrices are symmetric? Symbols $*$ and $\star$ represent real numbers.

$$A = [*] \quad 1\times 1 \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = B^T \qquad C = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} = C^T$$

$$D = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 2 & 0 & 7 & 4 \\ 0 & 7 & 6 & 0 \\ 1 & 4 & 0 & 3 \end{bmatrix} = F^T$$

$$D^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \neq \qquad E^T = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \neq$$

- For $A \in \mathbb{R}^{n\times n}$, $A + A^T$ is symm.
- For $A \in \mathbb{R}^{m\times n}$, * A^TA is symm *.

$$(A^TA)^T = A^T \cdot (A^T)^T = A^T \cdot A$$

A^TA is Symmetric

A very common example: For **any** matrix A with columns $a_1, \ldots, a_n$,

$$A^TA = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \cdots & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} a_1^Ta_1 & a_1^Ta_2 & \cdots & a_1^Ta_n \\ a_2^Ta_1 & a_2^Ta_2 & \cdots & a_2^Ta_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^Ta_1 & a_n^Ta_2 & \cdots & a_n^Ta_n \end{bmatrix}}_{\text{Entries are the dot products of columns of } A}$$

<u>Recall</u> real vectors

- $u \cdot v = u^T \cdot v$

- For Complex vectors u, v,

$$u \cdot v = \overline{u^T} \cdot v$$

<u>Example</u>

$$u = \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad u \cdot u = \overline{\begin{bmatrix} i & 1 \end{bmatrix}} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -i & 1 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = (-i) \cdot i + 1$$

$$= 2$$

- For real A, x, y

$$(Ax) \cdot y = (Ax)^T \cdot y = x^T \cdot (A^T \cdot y) = x \cdot (A^T \cdot y)$$

If A is symm.

$$(Ax) \cdot y = x \cdot (Ay).$$

Symmetric Matrices and their Eigenspaces

Theorem

A is a symmetric matrix, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$ corresponding to two distinct eigenvalues. Then $\vec{v}_1$ and $\vec{v}_2$ are orthogonal.

More generally, eigenspaces associated to distinct eigenvalues are orthogonal subspaces.

Proof:

$$Av_1 = \lambda_1 v_1 \qquad \lambda_1 \neq \lambda_2$$
$$Av_2 = \lambda_2 v_2$$

$$(Av_1) \cdot v_2 = (\lambda_1 v_1) \cdot v_2 = \lambda_1 (v_1 \cdot v_2)$$

$$\parallel \qquad\qquad \parallel$$

$$v_1 \cdot (Av_2) = v_1 \cdot (\lambda_2 v_2) = \lambda_2 (v_1 \cdot v_2)$$

$$\lambda_1 (v_1 \cdot v_2) = \lambda_2 (v_1 \cdot v_2)$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \cdot (v_1 \cdot v_2) = 0$$

$$v_1 \cdot v_2 = 0$$

Example 1

matrix with Orthonormal col.

Symm.

$\boxed{\text{Diagonalize}}$ A using an orthogonal matrix. Eigenvalues of A are given.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda = -1, 1$$

Hint: Gram-Schmidt

$\lambda = -1$: $\quad E_{-1} = \text{Nul}(A+I) = \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$= \text{Nul}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda = 1$: $\quad E_1 = \text{Nul}(A-I) = \text{Nul}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$= \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad v_3 = \frac{1}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} — v_1^T — \\ — v_2^T — \\ — v_3^T — \end{bmatrix}$$

$\underline{P}$ orthogonal matrix.

$P^T P = I \Rightarrow P^{-1} = P^T$.

Spectral Theorem

Recall: If P is an orthogonal $n \times n$ matrix, then $P^{-1} = P^T$, which implies $A = PDP^T$ is diagonalizable and symmetric.

Theorem: Spectral Theorem

An $n \times n$ symmetric matrix A has the following properties.

1. All eigenvalues of A are real.
2. The dimenison of each eigenspace is full, that it's dimension is **equal to** it's algebraic multiplicity.
3. The eigenspaces are mutually orthogonal.
4. A can be diagonalized: $A = PDP^T$, where D is diagonal and P is orthogonal.

Proof (if time permits):

$$A = P \cdot D \cdot P^T$$

$$A^T = (P^T)^T \cdot D^T \cdot P^T = P \cdot D^T \cdot P^T, \quad D^T = D$$

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① If A is real symm. $\lambda^{\in \mathbb{C}}$ eigenvalue of A

then λ is real.

Proof $A v = \lambda v$ for some $v \in \mathbb{C}^n$, $v \neq 0$

$\lambda \in \mathbb{C}$

$$(\lambda v) \cdot v = Av \cdot v = \overline{(Av)^T} \cdot v$$

$$\bar{\lambda}\,(v \cdot v) = \bar{v}^T \cdot (\overline{A^T} \cdot v) = v \cdot (\overline{A^T} v)$$

$$= \underbrace{v \cdot (A v)} = v \cdot (\lambda v) = \lambda\,(v \cdot v)$$

$$\bar{\lambda}\,(v \cdot v) = \lambda\,(v \cdot v)$$

$$\underbrace{(\bar{\lambda} - \lambda)}_{= 0}\;\underbrace{(v \cdot v)}_{\neq 0} = 0$$

$$\lambda = \bar{\lambda} \Rightarrow \lambda \text{ is real} \quad \blacksquare$$

② A : real symm. E : eigenspace for λ

If $x \in E^\perp$ then $Ax \in E^\perp$.

Proof $x \in E^\perp \Leftrightarrow \underbrace{x \cdot y = 0}$ for all $y \in E$

$Ay = \lambda y$ $\qquad \lambda\,(x \cdot y) = 0$

$$x \cdot (\lambda y) = 0 = x \cdot (Ay) = Ax \cdot y$$

$\blacksquare$

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Spectral Theorem

A : real $n \times n$ symmetric

(i) Every eigenvalue is <u>real</u>.

(ii) A is orthogonally diagonalizable, that is,

$A = P \cdot D \cdot P^T$, P : orthogonal matrix.

$$= \underbrace{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}}_{\text{ONB for } \mathbb{R}^n} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} — & v_1^T & — \\ — & v_2^T & — \\ & \vdots & \\ — & v_n^T & — \end{bmatrix}$$

$$= \lambda_1 \underbrace{\boxed{v_1 \cdot v_1^T}}_{\in \mathbb{R}^{n\times n}} + \lambda_2 v_2 \cdot v_2^T + \cdots + \lambda_n v_n \cdot v_n^T$$

(Spectral Decomposition)

<u>Note</u> If $v \in \mathbb{R}^n$ is a unit vector,

① $v \cdot v^T$ $n \times n$ symmetric

($(v v^T)^T = (v^T)^T \cdot v^T = v \cdot v^T$)

② $\mathrm{Rank}(v \cdot v^T) = 1$

<u>Ex</u> $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ 2 \ 3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(In general, $\mathrm{Rank}(A) = 1 \Rightarrow A = v \cdot w^T$)

③ For $y \in \mathbb{R}^n$ $\quad (v \cdot v^T) y = v \cdot (\underline{v^T \cdot y}) = (v \cdot y) \cdot v$

$$= \mathrm{proj}_v(y)$$

Spectral Decomposition of a Matrix

Spectral Decomposition

Suppose A can be orthogonally diagonalized as

$$A = PDP^T = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}$$

Then A has the decomposition

$$A = \lambda_1 \vec{u}_1 \vec{u}_1^T + \cdots + \lambda_n \vec{u}_n \vec{u}_n^T = \sum_{i=1}^{n} \lambda_i \vec{u}_i \vec{u}_i^T$$

Each term in the sum, $\lambda_i \vec{u}_i \vec{u}_i^T$, is an $n \times n$ matrix with rank 1.

Example 2

Construct a spectral decomposition for A whose orthogonal diagonalization is given.

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = PDP^T$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$4 = \lambda_1$, $2 = \lambda_2$

$\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = v_1$, $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = v_2$

orthogonal

lenth 1

$$= \lambda_1 v_1 \cdot v_1^T + \lambda_2 v_2 v_2^T$$

$$= 4 \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= 4 \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Section 7.2 : Quadratic Forms

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Quadratic forms
2. Change of variables
3. Principle axes theorem
4. Classifying quadratic forms

Learning Objectives

1. Characterize and classify quadratic forms using eigenvalues and eigenvectors.
2. Express quadratic forms in the form $Q(\vec{x}) = \vec{x}^T A\vec{x}$.
3. Apply the principle axes theorem to express quadratic forms with no cross-product terms.

Motivating Question Does this inequality hold for all x, y?

$$x^2 - 6xy + 9y^2 \geq 0$$

x, y : Real

minimum

$$\underbrace{(x-3y)^2}_{} \geq 0$$

x=3
y=1

Follows from Diagonalization of Symm.

Quadratic Forms

Definition

A **quadratic form** is a function $Q : \mathbb{R}^n \to \mathbb{R}$, given by

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

$\shortparallel$

$Q_A(\vec{x})$

Matrix A is $n \times n$ and symmetric.

In the above, $\vec{x}$ is a vector of variables.

Example 1

Compute the quadratic form $\vec{x}^T A \vec{x}$ for the matrices below.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix}$$

$$Q_A(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 4x_1^2 + 3x_2^2$$

$$Q_B(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 & x_1 - 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= (4x_1 + x_2)x_1 + (x_1 - 3x_2) \cdot x_2$$

$$= 4x_1^2 + x_2 x_1 + x_1 \cdot x_2 - 3x_2^2$$

$$= 4x_1^2 + 2x_1 \cdot x_2 - 3x_2^2$$

$$a_{11} x_1 \cdot x_1 + \underbrace{1 \cdot x_2 \cdot x_1}_{a_{21} \cdot x_2 \cdot x_1} + \underbrace{1 \cdot x_1 \cdot x_2}_{a_{12} x_1 \cdot x_2} + a_{22} \cdot x_2 \cdot x_2$$

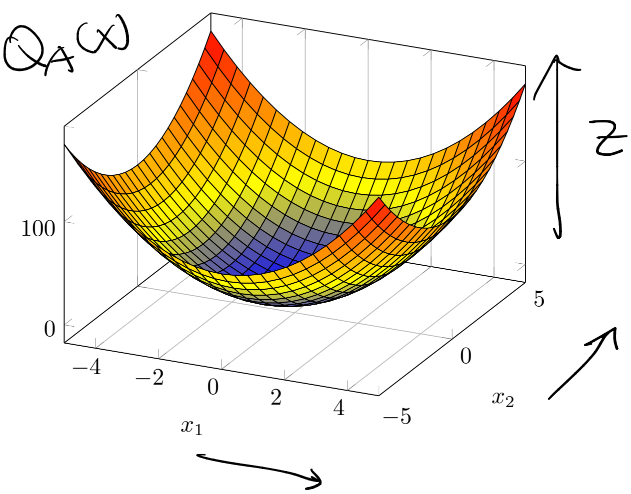
In general

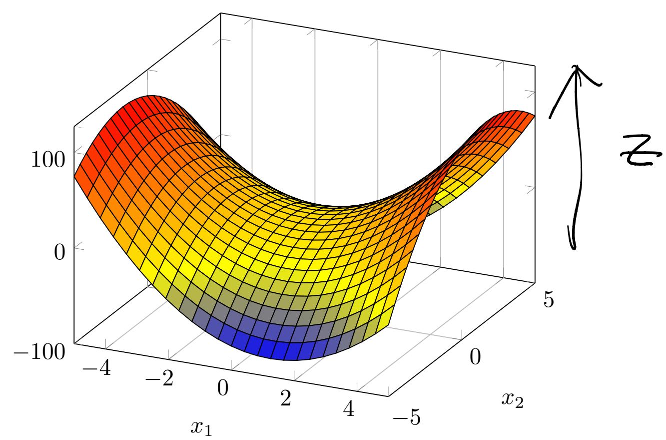
$$Q(x) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i \cdot x_j$$

Example 1 - Surface Plots

The surfaces for Example 1 are shown below.

$z = Q_A(x)$





Students are not expected to be able to sketch quadratic surfaces, but it is helpful to see what they look like.

$$Q_A(x) = 4x_1^2 + 3x_2^2 = z$$

$$Q_B(x) = 4x_1^2 + 2x_1x_2 - 3x_2^2 = z$$

Example 2

Write Q in the form $\vec{x}^T A\vec{x}$ for $\vec{x} \in \mathbb{R}^3$.

$$Q(x) = \underbrace{5x_1^2 - x_2^2 + 3x_3^2}_{} + \overbrace{6x_1x_3 - 12x_2x_3}^{\text{cross product.}} + 0 \cdot x_1 \cdot x_2$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \underbrace{\begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & -6 \\ 3 & -6 & 3 \end{bmatrix}}_{= A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Goal: $Q(x) = \oslash\,(y_1)^2 + \oslash\,(y_2)^2 + \cdots$

Remove cross product terms

Example

$$\begin{cases} y_1 = x_1 + 3x_2 \\ y_2 = -x_1 + 2x_2 \end{cases}$$

$$\begin{bmatrix} y_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$Q(x) = x^T \cdot A \cdot x \qquad A : \text{real symm.}$$

$$= \underbrace{\boxed{x^T \cdot P}}_{= y^T} \cdot D \cdot \underbrace{\boxed{P^T \cdot x}}_{= y}$$

$$y^T = (P^T x)^T = x^T \cdot P$$

$$= y^T \cdot D \cdot y$$

$$= [y_1 \ y_2 \ \cdots \ y_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 \cdot y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} — v_1^T — \\ — v_2^T — \\ \vdots \\ — v_n^T — \end{bmatrix} x = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{bmatrix}$$

$$= \lambda_1 (v_1 \cdot x)^2 + \cdots + \lambda_n (v_n \cdot x)^2$$

Change of Variable

If $\vec{x}$ is a variable vector in $\mathbb{R}^n$, then a **change of variable** can be represented as

$$\vec{x} = P\vec{y}, \quad \text{or} \quad \vec{y} = P^{-1}\vec{x}$$

With this change of variable, the quadratic form $\vec{x}^T A\vec{x}$ becomes:

Example 3

Make a change of variable $\vec{x} = P\vec{y}$ that transforms $Q = \vec{x}^T A \vec{x}$ so that it does not have cross terms. The orthogonal decomposition of A is given.

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} = PDP^T$$

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

v_1 v_2

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$$

$$y = P^T x$$

$$\begin{cases} y_1 = v_1 \cdot x \\ y_2 = v_2 \cdot x \end{cases}$$

$$Q(x) = 3x_1^2 + 4x_1 \cdot x_2 + 6x_2^2$$

$$= 2 \cdot y_1^2 + 7 \cdot y_2^2$$

$$= 2 \cdot \Big(\underbrace{\tfrac{1}{\sqrt{5}}(2x_1 - x_2)}_{\neq 0}\Big)^2 + 7\Big(\underbrace{\tfrac{1}{\sqrt{5}}(x_1 + 2x_2)}_{\neq 0}\Big)^2$$

$$\geq 0$$

$Q(x) = 0$ iff $\underline{y_1 = 0, \; y_2 = 0}$

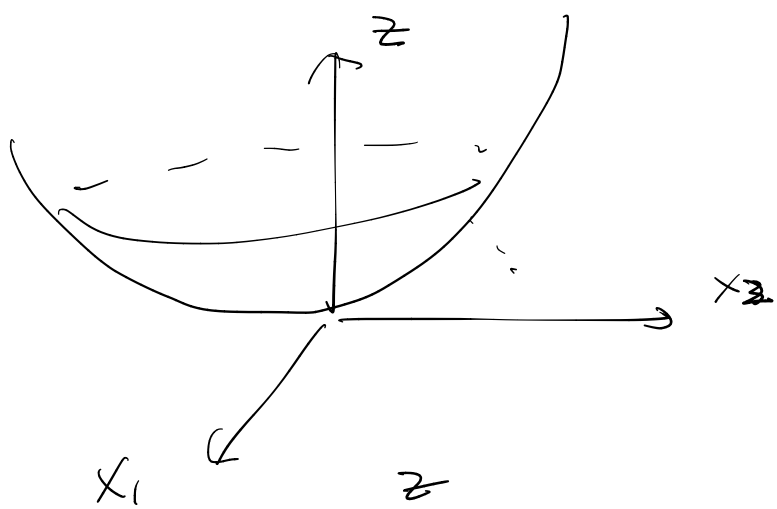
$\Rightarrow x_1 = 0 \quad x_2 = 0.$

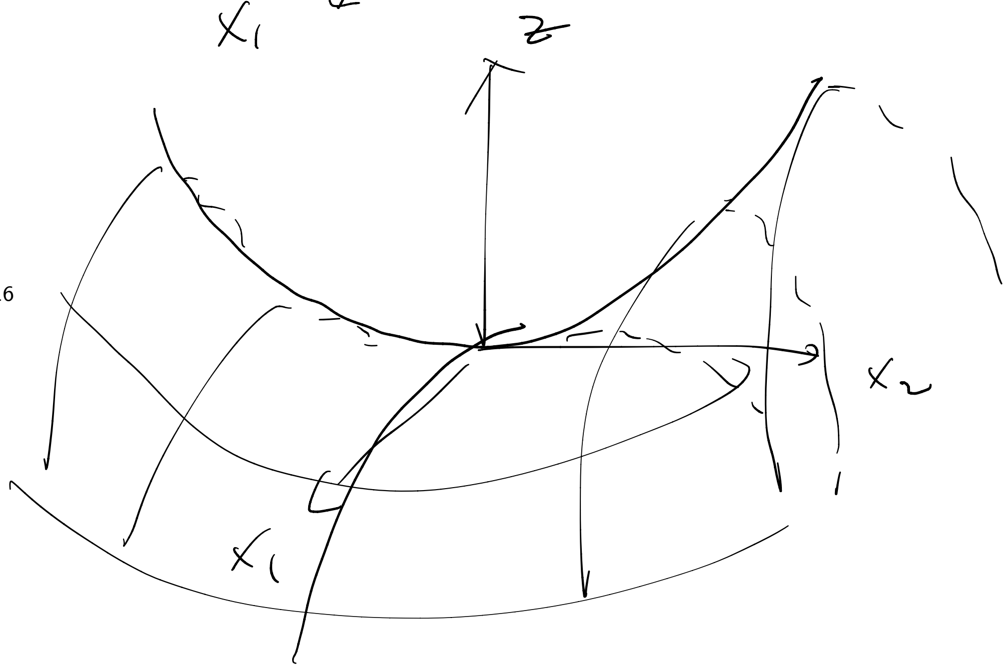
Geometry

Suppose $Q(\vec{x}) = \vec{x}^T A \vec{x}$, where $A \in \mathbb{R}^{n \times n}$ is symmetric. Then the set of $\vec{x}$ that satisfies

$$z \quad \cancel{c} = \vec{x}^T A \vec{x}$$

defines a curve or surface in $\mathbb{R}^n$.





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$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ & & \end{bmatrix}$$

$$Q(x) := x^T \cdot A \cdot x = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} \cdot x_i \cdot x_j$$

($x \in \mathbb{R}^n$, A: a real $n \times n$ symm. matrix)

By Spectral Thm

To remove cross product terms, use $A = P \cdot D \cdot P^T$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

eigenvectors & ONB for $\mathbb{R}^n$

$P^T \cdot P = I = P \cdot P^T$

$$Q(x) = x^T \cdot (P \cdot D \cdot P^T) \cdot x = \underbrace{(x^T \cdot P)}_{= y^T} \cdot \underset{\text{diagonal}}{D} \cdot \underbrace{(P^T \cdot x)}_{= y}$$

$$= \lambda_1 \cdot y_1^2 + \lambda_2 \cdot y_2^2 + \cdots + \lambda_n y_n^2 \qquad y = P^T x = \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{bmatrix}$$

$$= \lambda_1 (v_1 \cdot x)^2 + \lambda_2 (v_2 \cdot x)^2 + \cdots + \lambda_n (v_n \cdot x)^2 .$$

$$Q: \quad \underline{(v_1 \cdot x)^2 + (v_2 \cdot x)^2 + \cdots + (v_n \cdot x)^2} = \|x\|^2$$

$$= \|P^T \cdot x\|^2$$

(U has orthonormal columns, $\|x\| = \|Ux\|$)

$Ux \cdot Uy = x \cdot y$

Principle Axes Theorem

because P is orthogonal.

Theorem

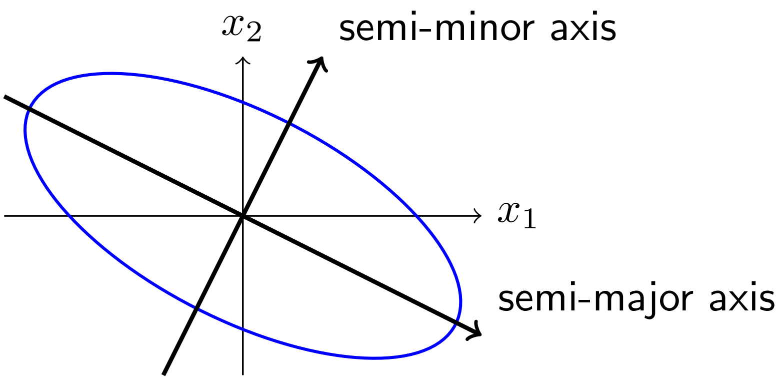
If A is a <u>real symm.</u> matrix then there exists an orthogonal change of variable $\vec{x} = P\vec{y}$ that transforms $\vec{x}^T A\vec{x}$ to $\vec{x}^T D\vec{x}$ with no cross-product terms.

Proof (if time permits):

$y = P^T x$

Example 5

Compute the quadratic form $Q = \vec{x}^T A \vec{x}$ for $A = \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, and find a change of variable that removes the cross-product term. A sketch of Q is below.



$Q(x) = x^T \cdot A \cdot x$ $\qquad A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$

λ : $\quad \det(A - \lambda I) = \lambda^2 - 13\lambda + 36$

$= (\lambda - 9)(\lambda - 4) = 0$

$\lambda = 9, 4$

$\lambda = 9$: $\quad Nul(A - 9I) = Nul \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} = Nul \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda = 4$: $\quad Nul(A - 4I) = Nul \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = Nul \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

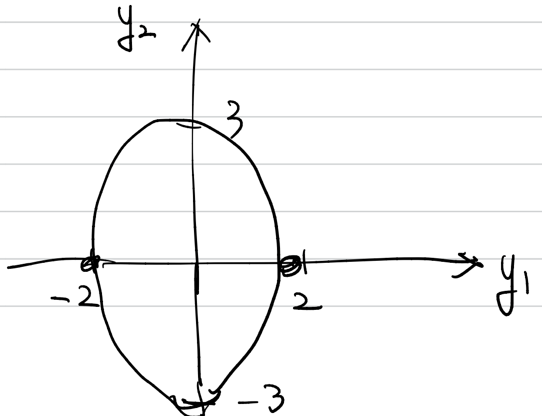
$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

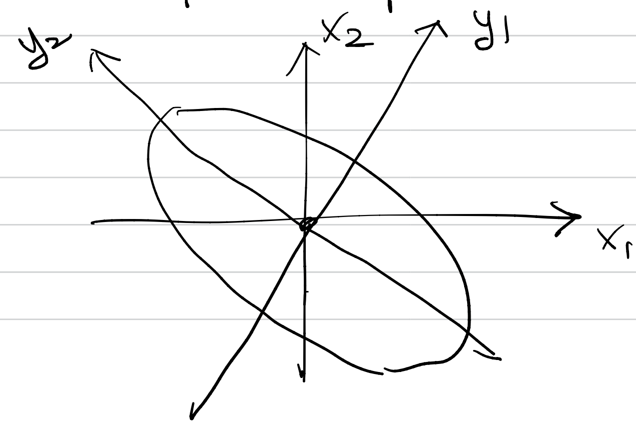
$Q(x) = 9 \cdot y_1^2 + 4 \cdot y_2^2 = 9(v_1 \cdot x)^2 + 4(v_2 \cdot x)^2$

$= 9 \cdot \left(\frac{1}{\sqrt{5}}(x_1 + 2x_2) \right)^2 + 4 \left(\frac{1}{\sqrt{5}}(2x_1 - x_2) \right)^2$

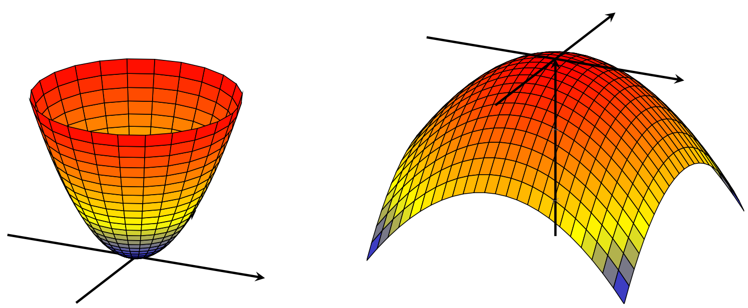
$\{ (x_1, x_2) : Q(x) = 36 \}$ $\qquad \dfrac{x_2 = -\frac{1}{2}x_1}{x_2 = 2x_1}$

$9y_1^2 + 4y_2^2 = 36$ $\qquad \frac{y_1^2}{4} + \frac{y_2^2}{9} = 1$





Classifying Quadratic Forms



Definition

A quadratic form Q is $Q(x) = 0$, if $x = 0$

1. **positive definite** if $Q(x) > 0$ for all $\vec{x} \neq \vec{0}$.
2. **negative definite** if $Q(x) < 0$ for all $\vec{x} \neq \vec{0}$.
3. **positive semidefinite** if $Q(x) \geq 0$ for all $\vec{x}$.
4. **negative semidefinite** if $Q(x) \leq 0$ for all $\vec{x}$.
5. **indefinite** if otherwise (meaning that $Q(x_1) > 0$, $Q(x_2) < 0$ for some x_1, x_2)

Note Q is Positive Definite if $\begin{cases} Q(x) \geq 0 & \text{all } x \\ Q(x) = 0 \text{ implies } x = 0. \end{cases}$

Quadratic Forms and Eigenvalues

Theorem

If A is a real $n\times n$ symm. matrix with eigenvalues λ_i, then $Q = \vec{x}^T A\vec{x}$ is

1. **positive definite** iff λ_i > 0 for all $i = 1, \dots, n$
2. **negative definite** iff λ_i < 0 for all $i = 1, \dots, n$
3. **indefinite** iff λ_i < 0, $\lambda_j > 0$ for some i, j

Proof (if time permits):

$$Q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 .$$

Example A: $m\times n$ real

$A^T A$: real $n\times n$ symm.

$$Q(x) = (x^T \cdot A^T)\cdot(A \cdot x) = (Ax)^T \cdot Ax$$

$$= \|Ax\|^2 \geq 0 \quad \text{for all } x.$$

$A^T A$ is positive semi definite.

Example 6

We can now return to our motivating question (from first slide): does this inequality hold for all x, y? $-3x\cdot y$ $\quad -3yx$

$$Q(x) = \underline{x^2 - 6xy + 9y^2 \ge 0}$$

$$A = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$ positive semidefinite.

$$\lambda = \ , \ \ge 0$$

$$\lambda^2 - 10\lambda + 0 = 0 \Rightarrow \lambda = 10, 0 \ge 0.$$

Section 7.3 : Constrained Optimization

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. Constrained optimization as an eigenvalue problem
2. Distance and orthogonality constraints

Learning Objectives

1. Apply eigenvalues and eigenvectors to solve optimization problems that are subject to distance and orthogonality constraints.

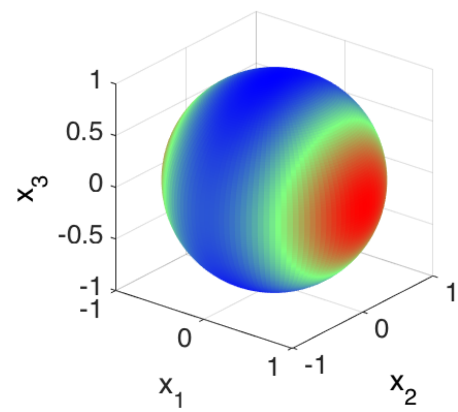
Example 1

The surface of a unit sphere in $\mathbb{R}^3$ is given by

$$1 = x_1^2 + x_2^2 + x_3^2 = ||\vec{x}||^2$$

Q is a quantity we want to optimize

$$Q(\vec{x}) = 9x_1^2 + 4x_2^2 + 3x_3^2$$



Find the largest and smallest values of Q on the surface of the sphere.

$$3 = 3(x_1^2 + x_2^2 + x_3^2) \leq Q(x) \leq 9(x_1^2 + x_2^2 + x_3^2) = 9$$

$$3 \leq Q \leq 9$$

max?

$Q(x) = 9$ if $x = \pm \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$Q(x) = 3$ if $x = \pm \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A Constrained Optimization Problem

Suppose we wish to find the maximum or minimum values of

$$Q(\vec{x}) = \vec{x}^{\,T} A \vec{x}$$

subject to

$$||\vec{x}|| = 1$$

That is, we want to find

$$m = \min\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$
$$M = \max\{Q(\vec{x}) : ||\vec{x}|| = 1\}$$

This is an example of a **constrained optimization** problem. Note that we may also want to know where these extreme values are obtained.

Constrained Optimization and Eigenvalues

Theorem

If $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated normalized eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Then, subject to the constraint $||\vec{x}|| = 1$,

- the **maximum** value of $Q(\vec{x}) = \lambda_1$, attained at $\vec{x} = \pm\vec{u}_1$.
- the **minimum** value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \pm\vec{u}_n$.

Proof:

Example 2

Calculate the maximum and minimum values of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$, and identify points where these values are obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

multiplicity $= 2$

$$\det(A - \lambda I) = -(\lambda-1)^2(\lambda+1) = 0 \qquad \lambda = 1, -1$$

$\lambda = 1$: $\quad \text{Nul}(A - I) = \text{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -1$: $\quad \text{Nul}(A + I) = \text{Nul}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad v_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$Q(x) = \lambda_1 \cdot (v_1 \cdot x)^2 + \lambda_2 (v_2 \cdot x)^2 + \lambda_3 (v_3 \cdot x)^2$$

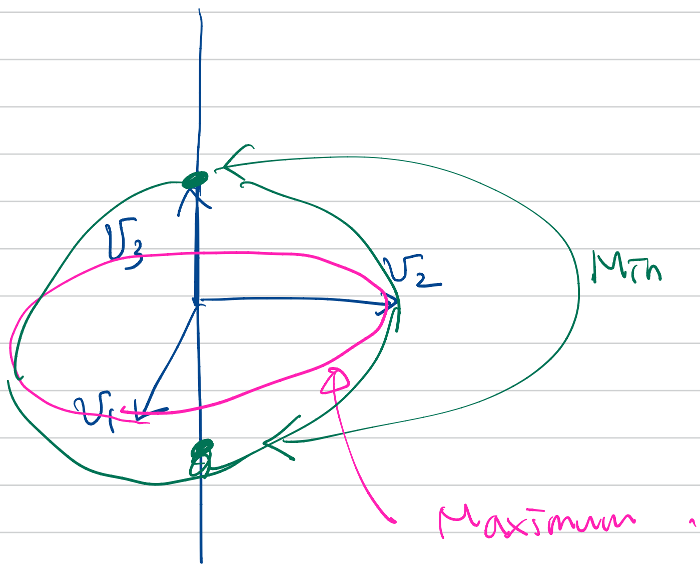
$$= (v_1 \cdot x)^2 + (v_2 \cdot x)^2 - (v_3 \cdot x)^2$$

$$\|x\|^2 = 1 = (v_1 \cdot x)^2 + (v_2 \cdot x)^2 + (v_3 \cdot x)^2$$

Max : $\underline{Q(x) = 1}$ when $x \cdot v_3 = 0$

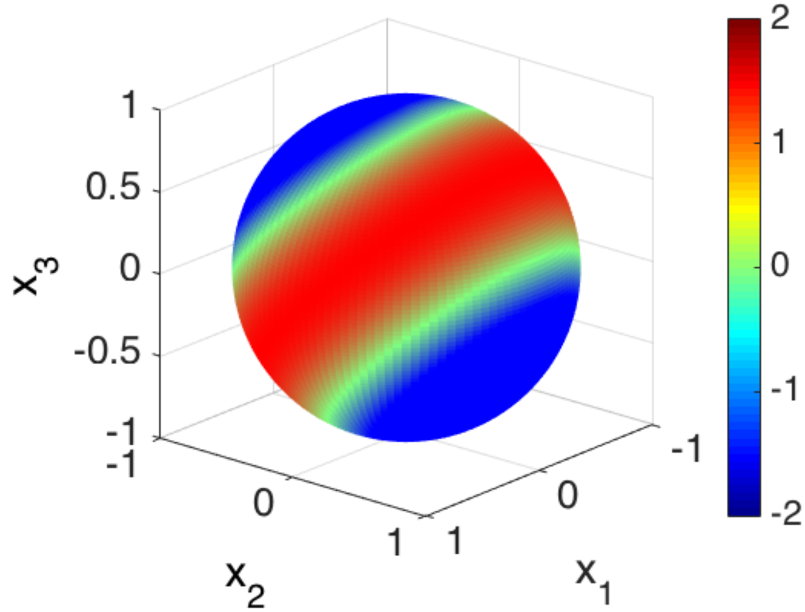
Min : $Q(x) = -1$ when $x \cdot v_1 = x \cdot v_2 = 0$.

$x = \pm v_3$



Example 2

The image below is the unit sphere whose surface is colored according to the quadratic from the previous example. Notice the agreement between our solution and the image.



An Orthogonality Constraint

Theorem

Suppose $Q = \vec{x}^T A\vec{x}$, A is a real $n \times n$ symmetric matrix, with eigenvalues

$$\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$$

and associated eigenvectors

$$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$$

Subject to the constraints $||\vec{x}|| = 1$ and $\vec{x} \cdot \vec{u}_1 = 0$,

- The maximum value of $Q(\vec{x}) = \lambda_2$, attained at $\vec{x} = \vec{u}_*$.
- The minimum value of $Q(\vec{x}) = \lambda_n$, attained at $\vec{x} = \vec{u}_n$.

Note that λ_2 is the second largest eigenvalue of A.

Example 3

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 1$ and to $\vec{x} \cdot \vec{u}_1 = 0$, and identify a point where this maximum is obtained.

$$Q(\vec{x}) = x_1^2 + 2x_2x_3, \qquad \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$Q(x) = 1 \cdot \cancel{(v_1 \cdot x)^2}^{\,0} + \boxed{1 \cdot (v_2 \cdot x)^2 + (-1) \cdot (v_3 \cdot x)^2}$$

$$\begin{cases} \|x\|^2 = 1 = \cancel{(v_1 \cdot x)^2}^{\,0} + \underbrace{\boxed{(v_2 \cdot x)^2 + (v_3 \cdot x)^2}}_{=1} \\ v_1 \cdot x = 0 \end{cases}$$

Max : $Q(x) = 1$ when $\begin{cases} x \cdot v_3 = 0 \\ x \cdot v_1 = 0 \end{cases}$

$\Downarrow$

$x = \pm v_2$

Example 4 (if time permits)

Calculate the maximum value of $Q(\vec{x}) = \vec{x}^T A\vec{x}$, $\vec{x} \in \mathbb{R}^3$, subject to $||\vec{x}|| = 5$, and identify a point where this maximum is obtained.

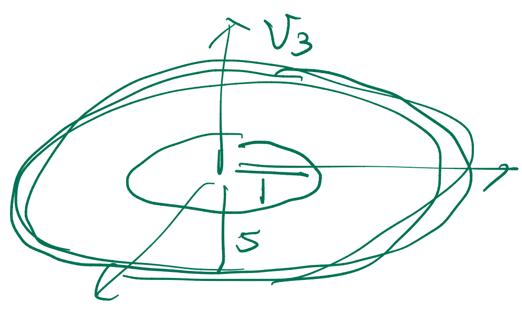
$$Q(\vec{x}) = x_1^2 + 2x_2x_3$$

Max = ? $\underline{1}$, 5, 25, $\frac{1}{5}$

$Q(x) = x^T \cdot A \cdot x$

Max = 25   when? $x = \pm \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} = \pm 5 \cdot v_1$

$x \cdot v_3 = 0$ & $||x|| = 5$



Min = −25   when $x = \pm 5 v_3$

Section 7.4 : The Singular Value Decomposition

Chapter 7: Orthogonality and Least Squares

Math 1554 Linear Algebra

Topics and Objectives

Topics

1. The Singular Value Decomposition (SVD) and some of its applications.

Learning Objectives

1. Compute the SVD for a rectangular matrix.
2. Apply the SVD to
   - estimate the rank and condition number of a matrix,
   - construct a basis for the four fundamental spaces of a matrix, and
   - construct a spectral decomposition of a matrix.

<u>Recall</u> (Spectral Decomposition)

A : $n \times n$ real symm.

Then (i) Every eigenvalue is real.

(ii) Diagonalizable $A = P \cdot D \cdot P^T$

$\{v_1, v_2, \cdots, v_n\}$: ONB for $\mathbb{R}^n$ eigenvector.

$\Downarrow$?

(iii) $A = \lambda_1 \underline{v_1 \cdot v_1^T} + \lambda_2 \underline{v_2 \cdot v_2^T} + \cdots + \lambda_n \underline{v_n \cdot v_n^T}$

$P = [v_1 \ v_2 \ \cdots \ v_n]$, $P^T P = I = P \cdot P^T$

$I = v_1 \cdot v_1^T + v_2 \cdot v_2^T + \cdots + v_n \cdot v_n^T$

$A \cdot I = \underline{A v_1} \cdot v_1^T + \underline{A v_2} \cdot v_2^T + \cdots + A v_n \cdot v_n^T$

$A = \lambda_1 v_1 \cdot v_1^T + \lambda_2 v_2 \cdot v_2^T + \cdots + \lambda_n v_n \cdot v_n^T$.

Now, A : $m \times n$ real

$A^T A$: $\underline{n \times n}$ real symm. positive semi definite (p.s.d).

$Q(x) = x^T \cdot (A^T \cdot A) \cdot x = (x^T \cdot A^T)(A \cdot x)$

$= (A \cdot x)^T (A x) = \|Ax\|^2 \geq 0$

Eigenvalues for $A^T A$

$\Rightarrow \begin{cases} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0 \\ \{v_1, v_2, \cdots, v_n\} : \text{ONB for } \mathbb{R}^n \end{cases}$

Eigenvectors for $A^T A$.

$\Rightarrow \underset{m \times n}{A} \underset{n \times n}{I} = A v_1 \cdot v_1^T + A v_2 \cdot v_2^T + \cdots + A v_n \cdot v_n^T$

① $v_{r+1}, v_{r+2}, \cdots, v_n$: eigenvectors for A^TA, $\lambda = 0$

$A^T \cdot A v_k = 0$ for $k = r+1, \cdots, n$

$0 = v_k^T \cdot A^T \cdot A \cdot v_k = (Av_k)^T \cdot (Av_k) = \|Av_k\|^2$

$Av_k = 0$

$A = Av_1 \cdot v_1^T + Av_2 \cdot v_2^T + \cdots + Av_r \cdot v_r^T$.

② $Col(A) = Span\{Ae_1, Ae_2, \cdots, Ae_n\}$

$e_1, \cdots, e_n$ can be written as lin. combi. of $v_1, \cdots, v_n$

$= Span\{\underline{Av_1, Av_2, \cdots, Av_r}\}$ orthogonal $\Rightarrow$ lin. indep.

$(Av_i) \cdot (Av_j) = v_i^T \cdot (\underbrace{A^T \cdot A \cdot v_j}_{= \lambda_j \cdot v_j}) = \lambda_j (v_i \cdot v_j)$

$= 0$ if $i \neq j$

$\|Av_i\|^2 = Av_i \cdot Av_i = \lambda_i \underbrace{(v_i \cdot v_i)}_{=1} = \lambda_i$

$\{\frac{1}{\sqrt{\lambda_1}} Av_1, \frac{1}{\sqrt{\lambda_2}} Av_2, \cdots, \frac{1}{\sqrt{\lambda_r}} Av_r\}$ Ortho~~gonal~~normal basis for $Col(A)$

$r = \dim(Col(A)) = Rank(A)$

<u>Def</u> $\sigma_i = \sqrt{\lambda_i}$: the singular value for A.

$u_i = \frac{1}{\sqrt{\lambda_i}} Av_i = \frac{1}{\sigma_i} Av_i$, $i = 1, 2, \cdots, r$

$\{u_1, u_2, \cdots, u_r\}$: ONB for $Col(A)$.

$A = \sigma_1 \cdot \frac{1}{\sigma_1} Av_1 \cdot v_1^T + \sigma_2 \cdot \frac{1}{\sigma_2} Av_2 \cdot v_2^T + \cdots + \sigma_r \cdot \frac{1}{\sigma_r} Av_r \cdot v_r^T$

$= \sigma_1 \cdot u_1 \cdot v_1^T + \sigma_2 \, u_2 \cdot v_2^T + \cdots + \sigma_r \cdot u_r \cdot v_r^T$.

: Singular Value Decomposition.

$\{v_1, v_2, \cdots, v_n\}$ Eigenvectors for $A^T A$
ONB for $\mathbb{R}^n$

$\{v_1, \cdots, v_r\}$: ONB for $Nul(A)^\perp = Col(A^T) = Row(A)$

$\{v_{r+1}, v_{r+2}, \cdots, v_n\}$: ONB for $Nul(A)$ $\left(= Nul(A^T A)\right)$

Eigenvectors for $\lambda = 0$ ($A^T A$)

$A v_k = 0$

$\{u_1, u_2, \cdots, u_r\}$: ONB for $Col(A)$ in $\mathbb{R}^m$

$\{u_{r+1}, \cdots, u_m\}$: ONB for $Col(A)^\perp$ in $\mathbb{R}^m$
(By Gram-Schmidt)

$$A = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma} \cdot \underset{n\times n}{V^T}$$

$U = [u_1 \; u_2 \; \cdots \; u_r \; u_{r+1} \; \cdots \; u_m]$ Orthogonal Mat.

$V = [v_1 \; v_2 \; \cdots \; v_n]$ Orthogonal Mat.

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & O & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & O & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}$$

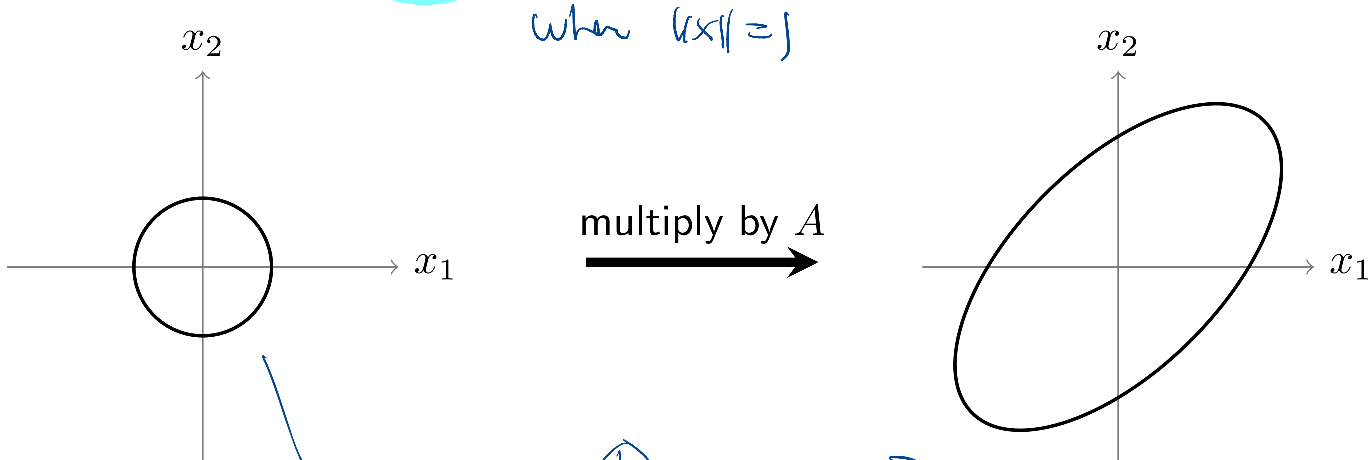
Example 1

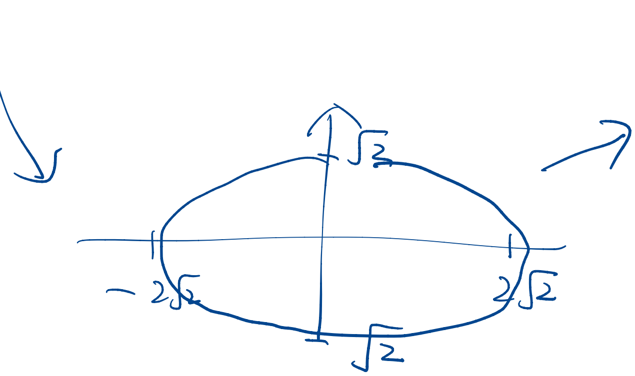
The linear transform whose standard matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}$$

maps the unit circle in $\mathbb{R}^2$ to an ellipse, as shown below. Identify the unit vector $\vec{x}$ in which $||A\vec{x}||$ is maximized and compute this length.

when $\|x\| = 1$





Max of $\|Ax\| = 2\sqrt{2}$ when $x = \pm\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

Min of $\|Ax\| = \sqrt{2}$ when $x = \pm\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$Q(x) = \|Ax\|^2 = (Ax)^T \cdot (Ax)$$

$$= x^T \cdot (A^T A) \cdot x.$$

Example 1 - Solution

Singular Values

Positive Semi Definite

The matrix A^TA is always symmetric, with non-negative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be the associated orthonormal eigenvectors. Then

ONB for $\mathbb{R}^n$

$$\|A\vec{v}_j\|^2 = \lambda_j$$

If the A has rank r, then $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ is an orthogonal basis for Col$A$: For $1 \leq j < k \leq r$:

$$(A\vec{v}_j)^T A\vec{v}_k = 0$$

$j \neq k$

Definition: $\sigma_1 = \sqrt{\lambda_1} \geq \sigma_2 = \sqrt{\lambda_2} \cdots \geq \sigma_n = \sqrt{\lambda_n}$ are the singular values of A.

The SVD

$$\underset{m\times n}{A} = \underset{m\times m}{U} \cdot \underset{m\times n}{\Sigma} \cdot \underset{n\times n}{V^T}$$

Theorem: Singular Value Decomposition

A $m \times n$ matrix with rank r and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ has a decomposition $U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}_{m\times n} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & \\ 0 & \sigma_2 & \dots & \vdots & \mathbf{0} \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \dots & \sigma_r & \\ & \mathbf{0} & & & \mathbf{0} \end{bmatrix}$$

U is a $m \times m$ orthogonal matrix, and V is a $n \times n$ orthogonal matrix.

① $V := [\underbrace{v_1, v_2, \dots, v_n}]$ ONB, Eigenvectors for $A^T \cdot A$

② Σ : $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots, \sigma_r = \sqrt{\lambda_r}$

$\lambda_{r+1} = \dots = \lambda_n = 0$

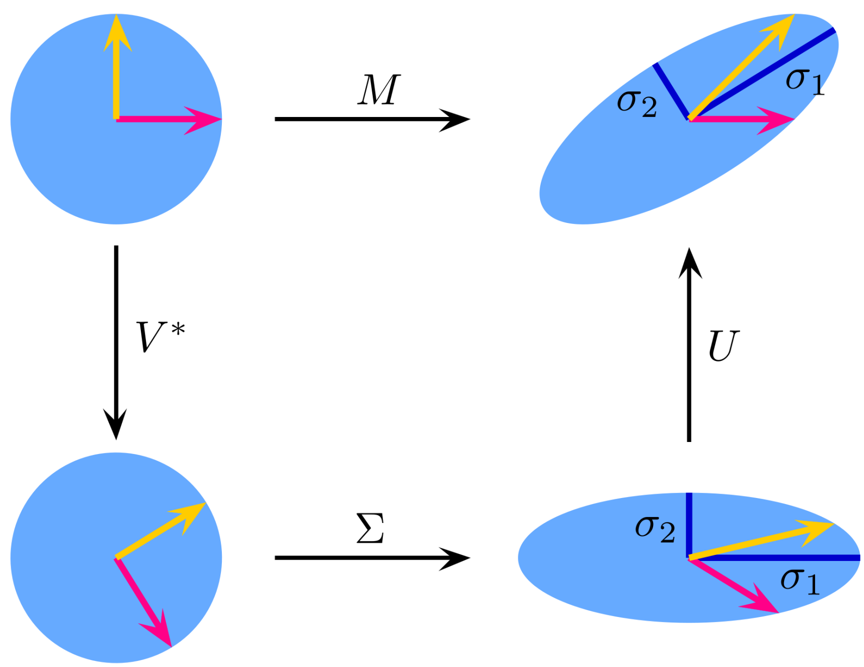
Eigenvalues for $A^T \cdot A$.

③ $U := [\underline{u_1, \dots, u_r}, u_{r+1}, \dots, u_m]$

$u_i = \frac{1}{\sigma_i} A v_i$

$i = 1, \ldots, r$

ONB for $\mathrm{Col}(A)^\perp$ using Gram-Schmidt.



$$M = U \cdot \Sigma \cdot V^*$$

Algorithm to find the SVD of A

Suppose A is $m \times n$ and has rank $r \leq n$.

1. Compute the squared singular values of $A^T A$, σ_i^2, and construct Σ.

2. Compute the unit singular vectors of $A^T A$, $\vec{v}_i$, use them to form V.

3. Compute an orthonormal basis for Col$A$ using

$$\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i, \quad i = 1, 2, \ldots r$$

   Extend the set $\{\vec{u}_i\}$ to form an orthonomal basis for $\mathbb{R}^m$, use the basis for form U.

Example 2: Write down the singular value decomposition for

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^T \cdot A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$\lambda_1 = 9$, $\lambda_2 = 4$ $\sigma_1 = 3$, $\sigma_2 = 2$

$r = n = 2$

$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$u_1 = \frac{1}{3} A v_1 = \frac{1}{3} \begin{bmatrix} 0 \\ -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{2} A v_2 = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 3: Construct the singular value decomposition of

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{3\times 2} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{2\times 2} = V^T$$

$r = 1$

(It has rank 1.)

$$A^TA = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

Eigenvalues

$$\det(A^TA - \lambda I) = \lambda^2 - 18\lambda + 0 = 0$$

$$\lambda_1 = \boxed{18}, \quad \lambda_2 = \boxed{0}$$

Eigenvector

$$\mathrm{Nul}(A^TA - 18I) = \mathrm{Nul}\begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

Similarly, $\boxed{v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ $\quad \underline{r = 1}$.

$$\sigma_1 = \sqrt{18}.$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{18}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 \\ -4 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$\{u_2, u_3\}$: ONB for $Col(A)^\perp = \underline{Nul(A^T)}$

$$Nul(A^T) = Nul \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Parametric Vector Form $\Rightarrow$ $\underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}_{\text{Basis for } Col(A)^\perp}$

$$u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{5}} \cdot (-4) \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{45}} \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

⟨Singular Value Decomposition⟩

A : $m \times n$ real

A^TA : $n \times n$ real symm. positive semidefinite.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0$$

$\{v_1, v_2, \cdots, v_n\}$ ONB for $\mathbb{R}^n$ eigenvectors.

$Av_k = 0$ for $k = r+1, r+2, \cdots, n$ ($\mathrm{Nul}(A) = \mathrm{Nul}(A^TA)$)

$$A = \sigma_1 \cdot \underbrace{\frac{1}{\sigma_1} A \cdot v_1}_{=u_1} \cdot v_1^T + \sigma_2 \underbrace{\frac{1}{\sigma_2} A v_2}_{=u_2} v_2^T + \cdots + \sigma_r \cdot \underbrace{\frac{1}{\sigma_r} A v_r}_{=u_r} \cdot v_r^T$$

$\sigma_i = \sqrt{\lambda_i}$: Singular values for A.

$$= \sigma_1 \cdot u_1 \cdot v_1^T + \sigma_2 u_2 \cdot v_2^T + \cdots + \sigma_r u_r \cdot v_r^T$$

$\{u_1, u_2, \cdots, u_r\}$: ONB for $\underline{\mathrm{Col}(A)}$

$\{u_{r+1}, \cdots, u_m\}$: ONB for $\mathrm{Col}(A)^\perp$ ($r = \mathrm{Rank}(A)$
$= \mathrm{Rank}(A^T \cdot A)$)

$\{v_{r+1}, v_{r+2}, \cdots, v_n\}$: ONB for $\mathrm{Nul}(A)$

eigenvector for $\lambda = 0$

in $\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)$

$\{v_1, v_2, \cdots, v_r\}$: ONB for $\mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T)$

$= \mathrm{Row}(A)$

$$\Rightarrow \underset{m\times n}{A} = \underset{m\times m}{U} \cdot \underset{m \times n}{\Sigma} \cdot \underset{n\times n}{V^T}, \quad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & & \ddots & \\ & 0 & & \sigma_r \\ & & & & 0 \end{bmatrix}$$

$U = [u_1, \cdots, u_m]$, $V = [v_1, \cdots, v_n]$

Applications of the SVD

The SVD has been applied to many modern applications in CS, engineering, and mathematics (our textbook mentions the first four).

- Estimating the rank and condition number of a matrix
- Constructing bases for the four fundamental spaces
- Computing the pseudoinverse of a matrix
- Linear least squares problems
- Non-linear least-squares
  https://en.wikipedia.org/wiki/Non-linear_least_squares
- Machine learning and data mining
  https://en.wikipedia.org/wiki/K-SVD
- Facial recognition
  https://en.wikipedia.org/wiki/Eigenface
- Principle component analysis
  https://en.wikipedia.org/wiki/Principal_component_analysis
- Image compression

Students are expected to be familiar with the 1st two items in the list.

The Condition Number of a Matrix

$$\underset{n\times n}{\Sigma} = \begin{bmatrix} \sigma_1 > 0 & & & \\ & \sigma_2 > 0 & & \\ & & \ddots & \\ & & & \sigma_n > 0 \end{bmatrix}$$

If A is an invertible $n \times n$ matrix, the ratio

$$\frac{\sigma_1 \leftarrow \text{largest}}{\sigma_n \leftarrow \text{smallest.}}$$

is the **condition number** of A.

Note that:

- The condition number of a matrix describes the sensitivity of a solution to $A\vec{x} = \vec{b}$ is to errors in A.
- We could define the condition number for a rectangular matrix, but that would go beyond the scope of this course.

$V^* = \overline{V}^T$

Example 4

ONB for Col(A)

For $A = U\Sigma V^*$, determine the rank of A, and orthonormal bases for Null$A$ and $(\text{Col}A)^\perp$.

rank(A)=3

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

ONB for Null (A)

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ONB for $\text{Col}(A)^\perp$

$= \{ u_4 \}$

$$\mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

ONB for $\text{Nul}(A)^\perp$

Example 4 - Solution

The Four Fundamental Spaces

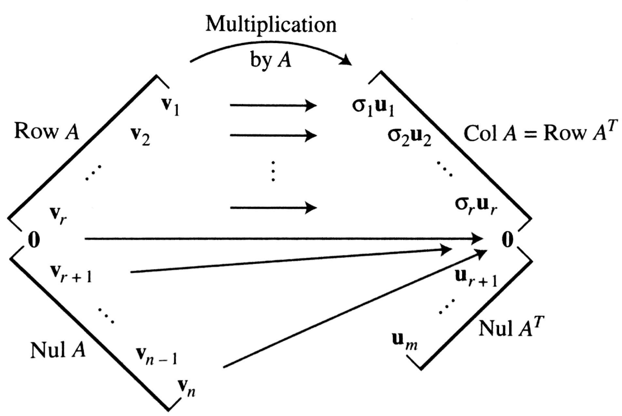


FIGURE 4 The four fundamental subspaces and the action of A.

1. $A\vec{v}_s = \sigma_s \vec{u}_s$.
2. $\vec{v}_1, \ldots, \vec{v}_r$ is an orthonormal basis for Row$A$.
3. $\vec{u}_1, \ldots, \vec{u}_r$ is an orthonormal basis for Col$A$.
4. $\vec{v}_{r+1}, \ldots, \vec{v}_n$ is an orthonormal basis for Null$A$.
5. $\vec{u}_{r+1}, \ldots, \vec{u}_n$ is an orthonormal basis for NullA^T.

The Spectral Decomposition of a Matrix

The SVD can also be used to construct the spectral decomposition for any matrix with rank r

$$A = \sum_{s=1}^{r} \sigma_s \vec{u}_s \vec{v}_s^T,$$

where $\vec{u}_s, \vec{v}_s$ are the s^{th} columns of U and V respectively.

For the case when $A = A^T$, we obtain the same spectral decomposition that we encountered in Section 7.2.