Quiz 6 Solution

Math 461 S22

Moment Generating Function

Let X be a random variable. Suppose the moment generating function $M_X(t)$ of X exists and is finite for $-\infty < t < \infty$. Then $M''_X(0) = \mathbb{E}[X^2]$.

- (a) True
- (b) False

$\boxed{\text{Solution}}$ If $M_X(t)$ is well-defined near 0, then

$$M_X'(0) = \mathbb{E}X \quad \text{and} \quad M_X''(0) = \mathbb{E}X^2.$$

Compute Expectation by Conditioning

Let $X > 0$ be a random variable with $\mathbb{E}[X] = 1$ and $\mathbb{E}[X^2] = 7$.

The conditional distribution of Y given $X = x$ is an exponential random variable with parameter $1/x^2$.

Then, $\mathbb{E}[Y] = 7$.

- (a) True
- (b) False

$\boxed{\text{Solution}}$

$$\mathbb{E}[Y] = \mathbb{E}[\underbrace{\mathbb{E}[Y|X]}_{= X^2 \text{ because } Y|X=x \sim \text{Exp}(\frac{1}{x^2})}]$$

$$= \mathbb{E}[X^2] = 7.$$

Conditional Expectation

Let X, Y be two continuous random variables with joint density given by

$$f(x,y) = \frac{12}{y} e^{-3xy^4} \qquad \text{for } x > 0, y > 1$$

and 0 otherwise.

Compute

$$\mathbb{E}[X \mid Y = 1].$$

$\boxed{\text{Solution}}$

$$f_Y(y) = \int_0^\infty \frac{12}{y} e^{-3xy^4}\, dx = \frac{12}{y} \frac{1}{3y^4} \left[- e^{-3xy^4} \right]_0^\infty$$

$$= \frac{4}{y^5}$$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = 3y^4 e^{-3xy^4} \quad \text{if } x>0 \quad \text{for } y>1$$

That is, $X | Y=y \sim \text{Exp}(3y^4) \quad \therefore \quad \mathbb{E}[X|Y=y] = \frac{1}{3y^4}$

$$\mathbb{E}[X|Y=1] = \frac{1}{3}.$$

Covariance

Let $X_1, X_2, \ldots$ be independent random variables with common mean 0 and common variance 1. Set

$$Y_n = 6X_n + 4X_{n+1}, \quad n \geq 1.$$

Find $\text{Cov}(Y_2, Y_3)$.

$\boxed{\text{Solution}}$

$$\text{Cov}(Y_2, Y_3) = \text{Cov}(6X_2 + 4X_3, 6X_3 + 4X_4)$$

$$= 36 \underbrace{\text{Cov}(X_2, X_3)}_{=0} + 24 \underbrace{\text{Cov}(X_2, X_4)}_{=0}$$

$$+ 24 \text{Cov}(X_3, X_3) + \underbrace{16 \text{Cov}(X_3, X_4)}_{=0}$$

$$= 24 \cdot \underbrace{\text{Var}(X_3)}_{=1.} = 24.$$