Homework 2 Solution

Math 461: Probability Theory, Spring 2022
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Due date: Feb 4, 2022

1. (a) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n = k.$$

(b) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each x_i is either 0 or 1 and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

Solution: (a) This is equivalent to choosing k positions out of n positions where the value will be 1 and 0 for the rest. Thus the answer is $\binom{n}{k}$.
(b) The answer is

$$\sum_{i=0}^{k} \binom{n}{i}$$

as the sum can be any numbers between 0 and k.

2. (a) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n = k.$$

(b) Suppose $k \geqslant n$. How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each $x_i \geqslant 1$ is a positive integer and

$$x_1 + x_2 + \cdots + x_n = k.$$

(c) How many vectors $(x_1, x_2, \ldots, x_n)$ are there for which each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n \leqslant k.$$

Solution: (a) Consider the linear ordering of $n-1$ of letter a and k of letter b. Then define x_i by the number of letter b between the $(i-1)$-th a and i-th of a. Then x_i are nonnegative integers and the sum of them is k. Thus, the number of ways is $\binom{(n-1)+k}{k}$. (b) Let $y_i = x_i - 1$, then y_i are nonnegative integers and the sum of y_i is $k-n$. By (a), the number of ways is $\binom{(n-1)+(k-n)}{n-1}$. (c) This can be done in two ways:
(i): Number of vectors $(x_1, x_2, \ldots, x_n)$ where each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_n = i,$$

is $\binom{n+i-1}{i}$. So the answer is

$$\sum_{i=0}^{k} \binom{n+i-1}{i}.$$

(ii): This is same as number of vectors $(x_1, x_2, \ldots, x_n, x_{n+1})$ where each $x_i \geqslant 0$ is a non-negative integer and

$$x_1 + x_2 + \cdots + x_{n+1} = k,$$

which is

$$\binom{n+k}{k}$$

3. Consider the set S of numbers $\{1, 2, \ldots, n\}$. One can see that the number of subsets of S size k is $\binom{n}{k}$. Count the same number in a different way depending on how many subsets of size k have i as their highest numbered member, to give a proof of the following identity known as Fermat's combinatorial identity: For all integers $n \geqslant k$

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

Solution: Suppose we want to choose a group of k many numbers from the set of numbers 1 through n. Clearly the number of choices is $\binom{n}{k}$. Now we can count the number in a different way. The largest number, say i, in the group of selected numbers can be anything from k to n. Given the largest number i, the number of ways to choose the remaining numbers is $\binom{i-1}{k-1}$. Thus the total number of choices is

$$\sum_{i=k}^{n} \binom{i-1}{k-1}.$$

Thus we have,

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}.$$

4. (a) In how many ways can n identical balls be distributed into r bins such that each bin contains at least two balls. Assume that $n \geqslant 2r$.
(b) Do the same problem as in (a), but now each bin contains at least three balls and $n \geqslant 3r$.

Solution: (a) This is equivalent to in how many ways we can write $x_1 + x_2 + \cdots + x_r = n$ where $x_i \geqslant 2$ for all i.

Given a sequence $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_r = n$ and $x_i \geqslant 2$ for all i, write $y_i = x_i - 2$ for all i. Then we have $y_i \geqslant 0$ for all i and

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

Note that we can get back x_i's from y_i's by adding 2 to each of them. Thus the answer is same as in how many ways we can write

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

where $y_i \geqslant 0$ for all i. We did this in class and the answer is $\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}$.
(b) $\binom{n-3r+r-1}{r-1} = \binom{n-2r-1}{r-1}$.

5. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b+g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i-th position, $1 \leqslant i \leqslant b+g$, is a girl?

Solution: We can compute all permutations of the $b+g$ people that have a girl in the i-th spot as follows. We have g choices for the specific girl we place in the i-th spot. Once this girl is selected we have $b+g-1$ other people to place in the $b+g-1$ slots around this i-th spot. This can be done in $(b+g-1)!$ ways. So the total number of ways to place a girl at position i is $g \cdot (b+g-1)!$. Thus the probability of finding a girl in the i-th spot is given by $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$.

6. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

Solution: Number of ways to choose two cards (ordered) from the deck of 52 cards is $52 \cdot 51$. Among them number of ways in which one of them is an ace (there are 4 aces with one of 4 suits) and the other is either a ten, a jack, a queen, or a king (there are total 16 cards) is $4 \cdot 16 \cdot 2$, where the last 2 is for ordering the cards. Thus the answer is

$$\frac{4 \cdot 16 \cdot 2}{52 \cdot 51} = 0.048.$$

7. A card player is dealt a 13 card hand from a well-shuffled, standard deck of cards. What is the probability that the hand is void in at least one suit ("void in a suit" means having no cards of that suit)?
Hint: Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*).

Solution: We want the probability that a given hand of bridge is void in *at least* one suit which means the hand could be void in more than one suit out of four suits. Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*). Then the probability we want is $\mathbb{P}(\cup_{i=1}^4 E_i)$ which we can calculate by using the inclusion-exclusion identity and the symmetry of the suits, given in this case by

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} \mathbb{P}(E_i) - \sum_{i=1}^{3} \sum_{j>i} \mathbb{P}(E_i E_j) + \sum_{i=1}^{3} \sum_{j>i} \sum_{k>j} \mathbb{P}(E_i E_j E_k) \\ &= 4\,\mathbb{P}(E_1) - 6\,\mathbb{P}(E_1 E_2) + 4\,\mathbb{P}(E_1 E_2 E_3).\end{aligned}$$

Note there is no terms $\mathbb{P}(E_i E_j E_k E_l)$ (which is zero) since we must be dealt some cards. We have $\mathbb{P}(E_1) = \frac{\binom{39}{13}}{\binom{52}{13}}$. Similarly

$$\mathbb{P}(E_1 E_2) = \frac{\binom{26}{13}}{\binom{52}{13}} \text{ and } \mathbb{P}(E_1 E_2 E_3) = \frac{\binom{13}{13}}{\binom{52}{13}} = \frac{1}{\binom{52}{13}}.$$

Thus we get

$$\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{1}{\binom{52}{13}} \left(4\binom{39}{13} - 6\binom{26}{13} + 4 \right) = 0.051.$$

8. For a group of 10 people, find the probability that all 4 seasons (winter, spring, summer, fall) occur at least once each among their birthdays, assuming that all seasons are equally likely.
Hint: Let E_i be the event that there are no birthdays in the i-th season.

Solution: Let E_i be the event that there are no birthdays in the i-th season. The probability that all seasons occur at least once is $1 - \mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4)$. Note that $E_1 E_2 E_3 E_4 = \emptyset$;. Using the inclusion-exclusion principle and the symmetry of the seasons,

$$\begin{aligned}\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} \mathbb{P}(E_i) - \sum_{i=1}^{3} \sum_{j>i} \mathbb{P}(E_i E_j) + \sum_{i=1}^{3} \sum_{j>i} \sum_{k>j} \mathbb{P}(E_i E_j E_k) \\ &= 4\,\mathbb{P}(E_1) - 6\,\mathbb{P}(E_1 E_2) + 4\,\mathbb{P}(E_1 E_2 E_3).\end{aligned}$$

We have $\mathbb{P}(E_1) = (3/4)^{10}$. Similarly

$$\mathbb{P}(E_1 E_2) = \frac{1}{2^{10}} \text{ and } \mathbb{P}(E_1 E_2 E_3) = \frac{1}{4^{10}}.$$

Therefore, $\mathbb{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = 4(\frac{3}{4})^{10} - \frac{6}{2^{10}} + \frac{4}{4^{10}}$. So the probability that all four seasons occur at least once is $1 - \left(4(\frac{3}{4})^{10} - \frac{6}{2^{10}} + \frac{4}{4^{10}}\right) = 0.781$.

9. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

(d) Write out all the outcomes in the event AW.

Solution:

(a) There are 2^5 possible outcomes (two outcomes per component, use generalized counting principle).

(b) Represent an outcome as a binary number, e.g., 00101 means that components 3 and 5 work, and components 1, 2, and 4 do not work. Then

$$\begin{aligned} W =& \{11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111, \\ & 00110, 00111, 01110, 01111, 10110, 10111, 10101\}. \end{aligned}$$

(c) There are $2^3 = 8$ outcomes in A.

(d) $AW = \{11000, 11100\}$.

10. Consider an experiment that consists of determining the type of job – either blue-collar or white-collar – and the political affiliation – Republican, Democratic, or Independent – of the 20 members of an adult soccer team. How many outcomes are

(a) in the sample space?

(b) in the event that at least one of the team members is a blue-collar worker?

(c) in the event that none of the team members considers himself or herself an Independent?

Solution:

(a) There are 6^{20} outcomes in the sample space.

(b) There are 3^{20} outcomes without any blue-collar workers, so that there are $6^{20} - 3^{20}$ outcomes with at least one blue-collar worker.

(c) If there are no independents, then for each player, there are 4 outcomes, so that there are 4^{20} outcomes altogether.