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Conditioning for computing probabilities

Example $U \sim \text{Unif}(0,1)$ $\quad X \mid U = p \sim \text{Bin}(n,p)$

PMF of $X = ?$

$$p(k) = \mathbb{P}(X=k) = \mathbb{E}[\mathbb{1}_A] \quad , \quad A = \{X = k\}$$

$$= \mathbb{E}[\mathbb{E}[\mathbb{1}_A \mid U = p]]$$

$$= \int_0^1 \binom{n}{k} p^k (1-p)^{n-k} \, dp = \binom{n}{k} \text{Beta}(k+1, n-k+1)$$

$$= \binom{n}{k} \cdot \frac{\Gamma(k+1)\,\Gamma(n-k+1)}{\Gamma((k+1)+(n-k+1))} = \frac{\Gamma(n+1)}{\Gamma(n+2)} = \frac{1}{n+1}.$$

Moment Generating Functions

Def A MGF of X is defined by

$$M_X(t) = M(x) = \mathbb{E}[e^{tX}] = \begin{cases} \sum e^{tx} p(x) \\ \int e^{tx} f_X(x)\, dx \end{cases}$$

Example

① $X \sim \text{Bin}(n,p)$

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \cdot \mathbb{P}(X=k) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (e^t p + (1-p))^n$$

② $X \sim \text{Pois}(\lambda)$

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk}\, \mathbb{P}(X=k) = \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= \exp(\lambda(e^t - 1))$$

③ $X \sim N(0,1)$

$$M_X(t) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx = e^{\frac{t^2}{2}} \int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-t)^2} \, dx = e^{t^2/2}$$

Prop

(i) If $M_X(t)$ is well-defined near 0, ($\exists\, \varepsilon > 0$ s.t $M_X(t) < \infty$ for all $t \in (-\varepsilon, \varepsilon)$) then

$$M_X'(0) = \mathbb{E}X, \quad M''(0) = \mathbb{E}X^2, \cdots, M^{(n)}(0) = \mathbb{E}X^n.$$

(ii) X, Y indep $\iff$ $M_{X+Y}(t) = M_X(t)\, M_Y(t)$.

Application If $X \sim \mathrm{Pois}(\lambda_1)$, $Y \sim \mathrm{Pois}(\lambda_2)$ indep.

then $M_X(t) = \exp(\lambda_1(e^t - 1))$, $M_Y(t) = \exp(\lambda_2(e^t - 1))$.

Since $M_X(t) \cdot M_Y(t) = \exp\left((\lambda_1 + \lambda_2)(e^t - 1)\right) = M_{X+Y}(t)$,

we have $X + Y \sim \mathrm{Pois}(\lambda_1 + \lambda_2)$