Homework 5 Solution

Math 461: Probability Theory, Spring 2022
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1. Two teams are going to play a best-of-7 match (the match will end as soon as either team has won 4 games). Each game ends in a win for one team and a loss for the other team. Assume that each team is equally likely to win each game, and that the games played are independent. Find the mean and variance of the number of games played.

> **Solution**: Let X be the number of games played. Since the pmf is
>
> $$\begin{aligned} \mathbb{P}(X=4) &= 2\cdot(\frac{1}{2})^4 = \frac{2}{16} \\ \mathbb{P}(X=5) &= 2\cdot(\frac{1}{2})^5\binom{4}{1} = \frac{4}{16} \\ \mathbb{P}(X=6) &= 2\cdot(\frac{1}{2})^6\binom{5}{2} = \frac{5}{16} \\ \mathbb{P}(X=7) &= 2\cdot(\frac{1}{2})^7\binom{6}{3} = \frac{5}{16}, \end{aligned}$$
>
> the mean and the variance are
>
> $$\begin{aligned} \mathbb{E}[X] &= \frac{1}{16}(2\cdot 4+4\cdot 5+5\cdot 6+5\cdot 7) = \frac{93}{16} \\ \mathrm{Var}(X) &= \mathbb{E}[X^2]-(\mathbb{E}[X])^2 = \frac{557}{16}-\left(\frac{93}{16}\right)^2 = 1.02734... \end{aligned}$$

2. To determine whether they have a certain disease, 100 people are to have their blood tested. However, rather than testing each individual separately, it has been decided first to place the people into groups of 10. The blood samples of the 10 people in each group will be pooled and analyzed together. If the test is negative, one test will suffice for the 10 people, whereas if the test is positive, each of the 10 people will also be individually tested and, in all, 11 tests will be made on this group. Assume that the probability that a person has the disease is .1 for all people, independently of each other, and compute the expected number of tests necessary for each group. (Note that we are assuming that the pooled test will be positive if at least one person in the pool has the disease.)

> **Solution**: Let X be the number of tests needed for a group of ten people. Then $X=1$ or $X=11$, and $\mathbb{P}(X=1)=0.9^{10}=0.3487$ and $\mathbb{P}(X=11)=1-0.9^{10}=0.6513$. Hence $\mathbb{E}(X)=7.5132$.

3. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win $-$\$1.00. (That is, you lose \$1.00.) Calculate
(a) the expected value of the amount you win;
(b) the variance of the amount you win.

Solution: Let X be the win/loss after one game. Then $\mathbb{P}(X = 1.1) = \frac{2\binom{5}{2}}{\binom{10}{2}} = \frac{20}{45} = \frac{4}{9}$, and $\mathbb{P}(X = -1) = \frac{5}{9}$.

(a) $\mathbb{E}[X] = 1.1 \cdot \frac{4}{9} - \frac{5}{9} = -\frac{1}{15}$.

(b) $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.21 \cdot \frac{4}{9} + \frac{5}{9} - \frac{1}{225} = 1.0889$.

4. If $\mathbb{E}[X] = 1$ and $\mathrm{Var}(X) = 5$, find (a) $\mathbb{E}[(2 + X)^2]$ and (b) $\mathrm{Var}(4 + 3X)$.

Solution: Note that $\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = 5 + 1 = 6$.

(a) $\mathbb{E}[(2 + X)^2] = \mathbb{E}[4 + 4X + X^2] = 4 + 4\,\mathbb{E}[X] + \mathbb{E}[X^2] = 14$.

(b) $\mathrm{Var}(4 + 3X) = 9\,\mathrm{Var}(X) = 45$.

5. Let N be a nennegative integer-valued random variable. For nonnegative values a_j, $j \geqslant 1$, show that

$$\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)\,\mathbb{P}(N = j) = \sum_{i=1}^{\infty} a_i\,\mathbb{P}(N \geqslant i).$$

Then show that

$$\mathbb{E}[N] = \sum_{i=1}^{\infty} \mathbb{P}(N \geqslant i),$$
$$\mathbb{E}[N(N+1)] = 2\sum_{i=1}^{\infty} i\,\mathbb{P}(N \geqslant i).$$

Solution: We have

$$\begin{aligned}
\sum_{j=1}^{\infty}(a_1 + \cdots + a_j)\,\mathbb{P}(N = j) &= \sum_{j=1}^{\infty}\sum_{i=1}^{j} a_i\,\mathbb{P}(N = j) \\
&= \sum_{i=1}^{\infty} a_i \sum_{j=i}^{\infty} \mathbb{P}(N = j) \\
&= \sum_{i=1}^{\infty} a_i\,\mathbb{P}(N \geqslant i).
\end{aligned}$$

If $a_i = 1$ for all i, then

$$\begin{aligned}
\mathbb{E}[N] &= \sum_{j=1}^{\infty} j\,\mathbb{P}(N = j) \\
&= \sum_{j=1}^{\infty}(1 + \cdots + 1)\,\mathbb{P}(N = j) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(N \geqslant i).
\end{aligned}$$

If $a_i = 2i$ for all i, then

$$\begin{aligned}\mathbb{E}[N(N+1)] &= \sum_{j=1}^{\infty} j(j+1)\,\mathbb{P}(N=j)\\ &= \sum_{j=1}^{\infty} 2(1+2+\cdots+j)\,\mathbb{P}(N=j)\\ &= 2\sum_{i=1}^{\infty} i\,\mathbb{P}(N\geqslant i).\end{aligned}$$

6. Let $p > 0$ be fixed and X be a random variable with pmf

$$p_X(k) = c\binom{n}{k}p^k \text{ for } k = 0, 1, \dots, n.$$

Find the value of c. Identify the distribution of X and find its mean and variance.

Solution: We must have

$$\sum_{k=0}^{n} p_X(k) = 1.$$

Now

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} c\binom{n}{k}p^k = \sum_{k=0}^{n} c\binom{n}{k}p^k 1^{n-k} = c(1+p)^n.$$

Thus $c = (1+p)^{-n}$ and

$$p_X(k) = \binom{n}{k}\left(\frac{p}{1+p}\right)^k\left(1-\frac{p}{1+p}\right)^{n-k} \text{ for } k = 0, 1, \dots, n.$$

Thus $X \sim \mathrm{Bin}(n, \frac{p}{1+p})$. So, mean $E[X] = np/(1+p)$ and variance $\mathrm{Var}(X) = np/(1+p)^2$.

7. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.
(a) Find the probability that 3 or more accidents occur today.
(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

Solution: X is Poisson with parameter $\lambda = 3$.

(a) $\mathbb{P}(X \geqslant 3) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2) = 1 - e^{-3}\left(1+3+\frac{9}{2}\right) = 0.5768.$

(b) $\mathbb{P}(X \geqslant 3 | X \geqslant 1) = \frac{\mathbb{P}(X\geqslant 3)}{\mathbb{P}(X\geqslant 1)} = \frac{\mathbb{P}(X\geqslant 3)}{1-e^{-3}} = 0.6070.$

8. The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that, in 1000 hands of poker, you will be dealt at least 2 full houses.

Solution: Let X be Poisson with parameter $\lambda = 1000 \cdot 0.0014 = 1.4$. Then $\mathbb{P}(X \geqslant 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) = 1 - e^{-1.4}(1+1.4) = 0.4082$.

9. Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability $p_i, \sum_{i=1}^5 p_i = 1$.
(a) Find the expected number of boxes that do not have any balls.
(b) Find the expected number of boxes that have exactly 1 ball.

Solution: (a) For $i = 1, \ldots, 5$, let $X_i = 1$ if the i-th box is empty and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_5$ is the number of empty boxes. For $i = 1, \ldots, 5$,

$$\mathbb{E}\, X_i = \mathbb{P}(X_i = 1) = (1 - p_i)^{10}.$$

Thus

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_5) = \sum_{i=1}^{5} (1 - p_i)^{10}.$$

(b) For $i = 1, \ldots, 5$, let $Y_i = 1$ if the i-th box has exactly 1 ball and $Y_i = 0$ otherwise. Then $Y = Y_1 + \cdots + Y_5$ is the number of boxes that have exactly 1 ball. For $i = 1, \ldots, 5$,

$$\mathbb{E}(Y_i) = \mathbb{P}(Y_i = 1) = 10 p_i (1 - p_i)^9.$$

Thus

$$\mathbb{E}(Y) = \mathbb{E}(Y_1) + \cdots + \mathbb{E}(Y_5) = \sum_{i=1}^{5} 10 p_i (1 - p_i)^9.$$

10. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0, \\ b/4 & 0 \leqslant b < 1, \\ (b+1)/4 & 1 \leqslant b < 2, \\ 11/12 & 2 \leqslant b < 3, \\ 1 & 3 \leqslant b. \end{cases}$$

(a) Find $\mathbb{P}(X = i), i = 1, 2, 3$.
(b) Find $\mathbb{P}(1/2 < X < 3/2)$.

Solution: (a)

$$\mathbb{P}(X = 1) = \mathbb{P}(X \leqslant 1) - \mathbb{P}(X < 1) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$
$$\mathbb{P}(X = 2) = \frac{11}{12} - \frac{3}{4} = \frac{1}{6}$$
$$\mathbb{P}(X = 3) = 1 - \frac{11}{12} = \frac{1}{12}$$

(b) $\mathbb{P}(\frac{1}{2} < X < \frac{3}{2}) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$.