Practice Problems for Final

MATH 3215, Spring 2024

1. Let $X_1, X_2, \cdots, X_7$ be an i.i.d. sequence of Poisson random variables with parameter $\lambda = 2$. Let $W = \sum_{i=1}^{7} X_i$. Find the MGF of W. How is W distributed? Find the probabilities $\mathbb{P}(W = 6)$ and $\mathbb{P}(W = 5 | X_1 = 2)$.

$$M_W(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_7}(t) \quad \text{(by Independence)}$$

$$= (M_{X_1}(t))^7 \quad \text{(because identically distributed)}$$

$$= \left(e^{2(e^t - 1)} \right)^7 \quad \text{(because } X_1 \sim \text{Pois}(2) \text{)}$$

$$= e^{14(e^t - 1)}$$

Thus, $W \sim \text{Pois}(14)$.

$$\mathbb{P}(W = 6) = e^{-14} \frac{14^6}{6!}$$

$$\mathbb{P}(W = 5 \mid X_1 = 2) = \frac{\mathbb{P}(W = 5, X_1 = 2)}{\mathbb{P}(X_1 = 2)}$$

$$= \frac{\mathbb{P}(X_2 + \cdots + X_7 = 3, X_1 = 2)}{\mathbb{P}(X_1 = 2)}$$

Let $U = X_2 + \cdots + X_7$

then

$U \sim \text{Pois}(12)$

and

U is indep of X_1

$$= \frac{\mathbb{P}(U = 3)\mathbb{P}(X_1 = 2)}{\mathbb{P}(X_1 = 2)}$$

$$= \mathbb{P}(U = 3) = e^{-12} \frac{12^3}{3!}$$

2. Suppose $X \sim N(1,4)$ and $Y \sim N(2,12)$ are independent normal random variables. Let $W = X + Y$. Find the MGF of W. Find the probability $\mathbb{P}(3 \leq W \leq 9)$.

$$M_W(t) = M_X(t) \cdot M_Y(t) \quad (\text{by Independence})$$

$$= e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \, e^{\mu_2 t \; \frac{\sigma_2^2 t^2}{2}} \qquad (\mu_1 = 1,\ \sigma_1^2 = 4,\ \mu_2 = 2,\ \sigma_2^2 = 12)$$

$$= e^{3t + \frac{16}{2} \cdot t^2}$$

i.e. $\quad W \sim N(3, 16)$

$$\mathbb{P}(3 \leq W \leq 9) = \mathbb{P}\left(\frac{3-3}{4} \leq \frac{W-3}{4} \leq \frac{9-3}{4}\right)$$

$$= \mathbb{P}(0 \leq Z \leq 1.5) \quad (\text{here } Z \sim N(0,1))$$

$$= \Phi(1.5) - \frac{1}{2}$$

3. An instructor has 50 exams that will be graded in sequence. The times required to grade the 50 exams are independent, with a common distribution that has mean 20 minutes and standard deviation 4 minutes. Approximate the probability that the instructor will grade at least 25 of the exams in the first 450 minutes of work.

Let X_i be the time required to grade the i-th exam paper.

Then $X_1, X_2, \cdots, X_{50} \sim$ i.i.d with $\mu = 20$, $\sigma = 4$.

$\mathbb{P}$(At least 25 exams graded in 450 min)

$= \mathbb{P}$(Time to grade 25 exams $\leq$ 450 min)

$= \mathbb{P}(X_1 + \cdots + X_{25} \leq 450)$

By the CLT, for $\overline{X} = \frac{1}{25}(X_1 + \cdots + X_{25})$,

$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\overline{X} - 20}{4/5}$ is approximately $N(0,1)$.

Thus,

$$\mathbb{P}(X_1 + \cdots + X_{25} \leq 450) = \mathbb{P}\left(\overline{X} \leq \frac{450}{25}\right)$$

$$= \mathbb{P}\left(\frac{\overline{X} - 20}{4/5} \leq \frac{18 - 20}{4/5}\right)$$

$$\approx \mathbb{P}\left(Z \leq -\frac{5}{2}\right) = 1 - \Phi(2.5)$$

4. A certain type of electrical motors is defective with probability 1/100. Pick 1000 motors and let X be the number of defective ones among these 1000 motors.

   (a) What is the probability that among the 1000 motors 13 or less are defective, i.e., what is $\mathbb{P}(X \leq 13)$.

   (b) Using a normal approximation, with mid-point correction, write down an expression for the probability that among the 1000 mortors 13 or less are defective. Use the corresponding tables (and $\sqrt{\frac{99}{10}} \approx 3.15$) to find an approximate value for thsi probability

(a): Let X be the number of defective ones among 1000 moters.

Assuming Independence, $X \sim \mathrm{Bin}(1000, \frac{1}{100})$

$$P(X \leq 13) = \sum_{k=0}^{13} \binom{1000}{k} \left(\frac{1}{100}\right)^k \left(\frac{99}{100}\right)^{1000-k}$$

(b) $$P(X \leq 13) = P(0 \leq X \leq 13)$$

$$\underset{\uparrow}{=} P(-0.5 < X < 13.5)$$

half-unit correction

By CLT,

$$\frac{X - np}{\sqrt{np(1-p)}} = \frac{X - 10}{\sqrt{\frac{99}{10}}} \approx N(0,1)$$

$$P(X \leq 13) = P\left(\frac{-10.5}{\sqrt{99/10}} < \frac{X-10}{\sqrt{99/10}} < \frac{3.5}{\sqrt{99/10}}\right)$$

$$\approx P(\quad < Z < \quad) = \cdots$$

5. A fair die will be rolled 720 times independently.

(a) What is the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively? That is, what is $\mathbb{P}(135 \le X \le 150)$? Write down the probability without using the tables and approximations.

(b) Using a normal approximation, **without mid-point correction**, write down an expression for the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively. Use the corresponding tables to find an approximate value for this probability.

(c) Using a normal approximation, **with mid-point correction**, write down an expression for the probability that among the 720 rolls the number 6 will appear between 135 and 150 times inclusively. Use the corresponding tables to find an approximate value for this probability.

(a) X = the number 6 will appear among 720 rolls

Then, $X \sim \text{Bin}(720, \frac{1}{6})$

$$P(135 \le X \le 150) = \sum_{k=135}^{150} \binom{720}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{720-k}$$

(b) By the CLT, $\frac{X - np}{\sqrt{npq}} = \frac{X - 120}{\sqrt{100}} \approx N(0,1)$

$$P(135 \le X \le 150) = P\left(\frac{135-120}{10} \le \frac{X-120}{10} \le \frac{150-120}{10}\right)$$

$$\approx P(1.5 \le Z \le 3)$$

$$= \Phi(3) - \Phi(1.5)$$

(c) $P(135 \le X \le 150) = P(134.5 < X < 150.5)$

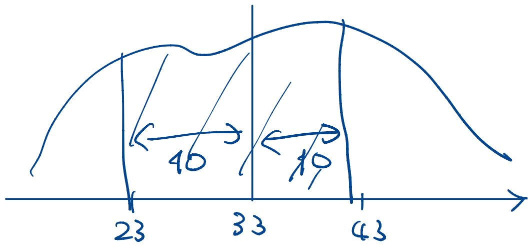
$$\approx P(1.45 < Z < 3.05)$$

$$= \Phi(3.05) - \Phi(1.45)$$

6. If X is a random variable with mean 33 and variance 16, use Chebyshev's inequality to find

(a) A lower bound for $\mathbb{P}(23 < X < 43)$.

(b) An upper bound for $\mathbb{P}(|X - 33| \geq 14)$.



$\mu = 33$

$\sigma^2 = 16$

(a): $\mathbb{P}(23 < X < 43) = \mathbb{P}(|X - 33| < 10)$

$= 1 - \mathbb{P}(|X - 33| \geq 10)$

$\geq 1 - \dfrac{\sigma^2}{10^2}$ (Chebyshev)

$= \dfrac{84}{100}$.

(b): $\mathbb{P}(|X - 33| \geq 14) \leq \dfrac{\sigma^2}{14^2} = \left(\dfrac{2}{7}\right)^2 = \dfrac{4}{49}$

7. Let $\overline{X}$ be the mean of a random sample of size $n = 15$ from a distribution with mean $\mu = 80$ and variance $\sigma^2 = 60$. Use Chebvshev's inequality to find a lower bound for $\mathbb{P}(75 < \overline{X} < 85)$.

$$\mathbb{E}[\overline{X}] = \mu = 80 \qquad \operatorname{Var}(\overline{X}) = \frac{\sigma^2}{n} = \frac{60}{15} = 4$$

$$\mathbb{P}(75 < \overline{X} < 85)$$

$$= \mathbb{P}(|\overline{X} - \mu| < 5)$$

$$= 1 - \mathbb{P}(|\overline{X} - \mu| \geqslant 5)$$

$$\geqslant 1 - \frac{\operatorname{Var}(\overline{X})}{5^2} = 1 - \frac{4}{25} = \frac{21}{25}.$$

$X_1, X_2, \cdots, X_{10}$

8. Let $W_1 < W_2 < \cdots < W_{10}$ be the order statistics of n independent observations from a $U(0,1)$ distribution.

   (a) Find the PDFs of W_1 and W_{10}.

   (b) Find $\mathbb{E}[W_1]$ and $\mathbb{E}[W_{10}]$.

(a):

$$F_{W_1}(t) = \mathbb{P}(W_1 \le t) = 1 - \mathbb{P}(W_1 > t) = 1 - (\mathbb{P}(X_1 > t))^{10}$$

$$= 1 - (1-t)^{10} \quad \text{for} \quad 0 < t < 1$$

$$f_{W_1}(t) = \begin{cases} 10(1-t)^9 & \text{for} \quad 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{W_{10}}(t) = \mathbb{P}(W_{10} \le t) = (\mathbb{P}(X_1 \le t))^{10}$$

$$= t^{10} \quad \text{for} \quad 0 < t < 1$$

$$f_{W_{10}}(t) = \begin{cases} 10\, t^9 & \text{for} \quad 0 < t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(b):

$$\mathbb{E}[W_1] = \int_0^1 t \cdot 10(1-t)^9 \, dt = \int_0^1 10\, t^9 (1-t)\, dt$$

$$= \left[t^{10} - \frac{10}{11} t^{11} \right]_0^1 = \frac{1}{11}$$

$$E[W_{10}] = \int_0^1 t \cdot 10 \cdot t^9 \, dt = \left[\frac{10}{11} t^{11} \right]_0^1 = \frac{10}{11}.$$

9. Let $Y_1 < Y_2 < \cdots < Y_5$ be the order statistics of a random sample of size 5 from a distribution with PDF $f(x) = e^{-x}$ for $0 < x < \infty$.

$X \sim \text{Exp}(1)$

(a) Find the PDF of Y_3.

(b) Find the PDF of $U = e^{-Y_3}$.

$$F_{Y_3}(t) = P(Y_3 \leq t)$$

$$= P(Y_5 \leq t) + P(Y_4 \leq t < Y_5) + P(Y_3 \leq t < Y_4)$$

$$= (P(X \leq t))^5 + \binom{5}{1} (P(X \leq t))^4 \, P(X > t) + \binom{5}{2} (P(X \leq t))^3 \, P(X > t)^2$$

$$= (1 - e^{-t})^5 + 5(1 - e^{-t})^4 e^{-t} + 10(1 - e^{-t})^3 e^{-2t}$$

$$f_{Y_3}(t) = \cancel{5(1-e^{-t})^4} \cdot e^{-t} + \cancel{20(1-e^{-t})^3} \cdot e^{-2t} - \cancel{5(1-e^{-t})^4} \cdot e^{-t}$$

$$+ 30(1 - e^{-t})^2 e^{-3t} - \cancel{20(1-e^{-t})^3} e^{-2t}$$

$$= 30(1 - e^{-t})^2 e^{-3t} \quad \text{for } t > 0$$

$$F_U(t) = P(U \leq t) = P(e^{-Y_3} \leq t)$$

$$= P(-Y_3 \leq \ln t)$$

$$= P(Y_3 \geq -\ln t) = 1 - P(Y_3 \leq -\ln t) = 1 - F_{Y_3}(-\ln t)$$

$$f_U(t) = -f_{Y_3}(-\ln t) \cdot \left(-\frac{1}{t}\right)$$

$$= 30(1 - e^{-(-\ln t)})^2 e^{-3(-\ln t)} \cdot \frac{1}{t}$$

$$= 30(1 - t)^2 t^3 \cdot \frac{1}{t} = \underline{30 t^2 (1 - t)^2} \; //$$

10. Suppose that X is a discrete random variable with pmf

$$f(x) = \frac{2+\theta(2-x)}{6}, \qquad x = 1, 2, 3,$$

where the unknown parameter θ belongs to the parameter space $\Omega = \{-1, 0, 1\}$. Suppose further that a random sample X_1, X_2, X_3, X_4 is taken from this distribution, and the four observed values are $(x_1, x_2, x_3, x_4) = (3, 2, 3, 1)$. Find the maximum likelihood estimate of θ.

Likelihood Function

$= \text{joint PMF} = f(x_1)\, f(x_2)\, f(x_3)\, f(x_4)$

$$= \left(\frac{2+\theta(2-3)}{6}\right)\left(\frac{2+\theta(2-2)}{6}\right)\left(\frac{2+\theta(2-3)}{6}\right)\left(\frac{2+\theta(2-1)}{6}\right)$$

$$= \frac{1}{6^4} \cdot (2-\theta)^2 \cdot 2 \cdot (2+\theta)$$

Let $g(\theta) = (2-\theta)^2 \cdot (2+\theta)$

Maximize $g(\theta)$ in $\theta \in \{-1, 0, 1\}$

$g(-1) = 9$

$g(0) = 8$

$g(1) = 3$

$\therefore \; \underline{\underline{\hat{\theta} = -1}} \leftarrow$ MLE.

11. A random sample of size 16 from the normal distribution $N(\mu, 25)$ yielded $\overline{X} = 73.8$. Find a 95% confidence interval for μ.

$$\left[\overline{X} - 1.96 \frac{\sigma}{\sqrt{n}} , \quad \overline{X} + 1.96 \frac{\sigma}{\sqrt{n}} \right]$$

$$= \left[73.8 - \underbrace{1.96 \cdot \frac{5}{\sqrt{16}}}_{=2.45} , \quad 73.8 + 1.96 \cdot \frac{5}{\sqrt{16}} \right]$$

$$= [71.35 , \; 76.25].$$