Final Take-Home Exam

Math 461: Probability Theory, Spring 2021
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Due date: May 12, 2021

1. Let X be a uniform random variable on $(-1,1)$ and $Y = |X|$.
   (a) Show that Y is also a uniform random variable.
   (b) Determine whether X and Y are independent.
   (c) Compute $\mathrm{Cov}(X,Y)$.

2. Let X, Y be discrete random variables defined by, for some $p_i, q_i > 0$ for $i = 1, 2, \cdots, n$

$$p_i = \mathbb{P}(X = i), \quad q_i = \mathbb{P}(Y = i), \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} q_i = 1.$$

   (a) The entropy of X is defined by

$$H(X) = -\sum_{i=1}^{n} p_i \log p_i.$$

   Show that if $q_1 = q_2 = \cdots = q_n = \frac{1}{n}$, then $H(X) \leqslant H(Y)$. In other words, the entropy of X is maximized when X is uniform. (Hint: Consider two random variables Z, W defined by

$$\mathbb{P}(Z = \frac{1}{p_i}) = p_i, \quad \mathbb{P}(W = \frac{1}{q_i}) = q_i,$$

   for $i = 1, 2, \cdots, n$. Then, use Jensen's inequality for $f(t) = \log t$.)

   (b) The Kullback–Leibler divergence between X and Y is defined by

$$D(X||Y) = \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right).$$

   Show that $D(X||Y) \geqslant 0$. (Hint: Consider a random variable Z defined by

$$\mathbb{P}(Z = \frac{p_i}{q_i}) = q_i,$$

   for $i = 1, 2, \cdots, n$. Then, use Jensen's inequality for $f(t) = t \log t$.)

3. Let X and Y be random variables with $0 < \mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$.
   (a) Show that $\mathbb{E}[XY]^2 \leqslant \mathbb{E}[X^2]\,\mathbb{E}[Y^2]$. (Hint: Consider $\mathbb{E}[(X - tY)^2] \geqslant 0$ and choose an appropriate value t to get the inequality.)
   (b) Assume that $X \geqslant 0$. Show that $\mathbb{P}(X > 0) \geqslant \frac{(\mathbb{E}[X])^2}{\mathbb{E}[X^2]}$. (Hint: Consider $Y = I_{\{X>0\}}$.)
   (c) Let $A_1, A_2, \cdots, A_n$ be events with

$$m = \sum_{i=1}^{n} \mathbb{P}(A_i), \qquad v = \sum_{i<j} \mathbb{P}(A_i \cap A_j).$$

   Show that

$$\frac{m^2}{m + 2v} \leqslant \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) \leqslant m.$$

   (Hint: Consider $X_i = I_{A_i}$, $X = \sum_{i=1}^{n} X_i$, and $Y = I_{\{X>0\}}$.)